Traffic grooming in a passive star WDM network

Eric Angel [1], Evripidis Bampis [1] and Fanny Pascual [1]

Abstract. We consider the traffic grooming problem in passive WDM star networks. Traffic grooming is concerned with the development of techniques for combining low speed traffic components onto high speed channels in order to minimize network cost. We first prove that the traffic grooming problem in star networks is NP-hard for a more restricted case than the one considered in [2]. Then, we propose a polynomial time algorithm for the special case where there are two wavelengths per fiber using matching techniques. Furthermore, we propose two reductions of our problem to two combinatorial optimization problems, the *constrained multiset multicover problem* [3], and the *demand matching problem* [4] allowing us to obtain a polynomial time H_{2C} (resp. $2 + \frac{4}{5}$) approximation algorithm for the minimization (resp. maximization) version of the problem, where C is the capacity of each wavelength.

Keywords: star, traffic grooming, WDM network, approximation, algorithm.

1 Introduction

Recently, in order to utilize bandwidth more effectively, new models appeared allowing several independent traffic streams to share the bandwidth of a lightpath. It is in general impossible to setup lightpaths between every pair of edge routers and thus it is natural to consider that traffic is electronically switched (groomed) onto new lightpaths toward the destination node. The introduction of electronic switching increases the degree of connectivity among the edge routers while at the same time it may reduce significantly wavelength requirements for a given traffic demand. The drawback of this approach is that the introduction of expensive active components (optical transceivers and electronic switches) may increase the cost of the network. These observations motivated R. Dutta and G.N. Rouskas [2] to study the traffic grooming problem that we consider in this paper in order to find a tradeoff between the cost of the network and its performance.

We focus on star networks composed by a set of transmitters, a set of receivers and a hub, and the goal is to minimize the *total amount of electronic switching*. This cost function measures exactly the amount of electronic switching inside the network (but it only indirectly captures the transceiver cost). Our interest to star networks besides their simplicity, which allows us to provide the first approximation algorithms with performance guarantee for this variant of the

[1] LaMI – Université d'Évry Val d'Essonne, CNRS UMR 8042 –, 523 Place des Terrasses, 91000 Évry, France. E-mail: {angel, bampis, fpascual}@lami.univ-evry.fr

traffic grooming problem, is also motivated by their use in the interconnection of LANs or MANs with a wide area backbone.

Problem definition

We consider a network in the form of a star with $N+1$ nodes. There is a single *hub* node which is connected to every other node by a physical link. All the nodes, except the hub, are divided into two groups V_1 and V_2: the nodes in V_1 are the transmitters and the nodes in V_2 are the receivers. The hub is numbered 0 and the N other nodes are numbered from 1 to N in some arbitrary order. Each physical link consists of a fiber, and each fiber can carry W wavelengths. Each wavelength has a capacity C, expressed in units of some arbitrary rate. We represent a traffic pattern by a demand matrix $T = [t_{ij}]$, where integer t_{ij} denotes the number of traffic streams (each unit demand) from node $i \in V_1$ to node $j \in V_2$. We do not allow the traffic demands to be greater than the capacity of a lightpath, i.e. for all $(i, j), 0 \leq t_{ij} \leq C$.

The hub has both optical and electronic switching capabilities: it let some lightpaths pass through transparently, while it may terminate some others. Traffic on terminated lightpaths is electronically switched (groomed) onto a new lightpath towards the destination node. A traffic demand (or request) t_{ij} must have its own wavelength from i to the hub and from the hub to j to be optically routed, whereas it can share a wavelength with some other traffic demands if it is electronically switched. The goal we consider in this paper is to *minimize the total amount of electronic switching at the hub*. This problem is often called *the traffic grooming problem.*

R. Dutta and G.N. Rouskas considered in [2] the traffic grooming problem in several network topologies, including a star network. However there are differences between their problem and ours: in [2], each node of the network, including the hub, can be a transmitter and a receiver, and traffic demands between two nodes are allowed to be greater than the capacity of a wavelength (i.e. it is possible that $t_{ij} > C$ for some i, j). To distinguish the two problems, we will call their problem the traffic grooming problem in an active star, and our problem *the traffic grooming problem in a passive star* (see Section 4 for an integer linear programming formulation of the problem). Once we know which traffic demands are optically routed, the wavelength assignment problem is easy in the case of a passive star network.

There are in fact two versions of the traffic grooming problem: either we want to minimize the total amount of electronic switching at the hub (this is the *minimization* version), or we want to maximize the total amount of traffic which is optically routed (this is the *maximization* version). These two versions are equivalent (i.e. an optimal solution for one is also an optimal solution for the other one) because the optimal solution of the maximization problem is equal to the sum of all the traffic demands, minus the optimal solution of the minimization problem.

Our results are as follows. First, we show in Section 2 that the traffic grooming problem in a passive star is NP-Complete, in both the minimization and the maximization versions of the problem. Then we show in Section 3 that these problems are polynomially solvable if there are only two wavelengths per fiber ($W = 2$): we give an algorithm which gives an optimal solution. In Section 4, we show that we cannot deduce a constant approximation guarantee of the maximization (resp. minimization) version from a constant approximation guarantee of the minimization (resp. maximization) version of the problem, and we give two approximation algorithms. The first one concerns the minimization version: we transform our problem in a constrained multiset multicover problem [3], and we get an approximation guarantee of H_{2C}. The second approximation algorithm concerns the maximization version: we transform our problem in a demand matching problem in a bipartite graph [4], and we obtain an approximation guarantee of $(2 + \frac{4}{5})$. We conclude the paper in Section 5.

2 Hardness results

Let us show in this section that the decision version of the grooming problem in a passive star is NP-Complete.

In order to do this proof, we were inspired by the proof of R. Dutta and G.N. Rouskas in [2]: in this paper they showed that their traffic grooming problem is NP-complete. They reduced the decision version of the Knapsack problem to their problem. We do the same reduction, replacing traffic demands t_{ij} greater than C by several traffic demands of the same weight from i, or to j. They also used traffic demands to the hub (or from the hub). We replace these traffic demands by traffic demands to some new nodes (or from some new nodes) and we force these traffic demands to be switched electronically at the hub.

We reduce the decision version of the Knapsack problem [1] to our grooming problem: let $Q \in Z^+$, is there a solution of our grooming problem in which the amount of optically routed traffic is greater or equal to Q? An instance of the Knapsack problem is given by a finite set U of cardinality n, for each element $u_i \in U$ a weight $w_i \in Z^+$, and a value $v_i \in Z^+, \forall i \in \{1, 2, ..., n\}$, a target weight $B \in Z^+$, and a target value $K \in Z^+$. The problem asks whether there exists a binary vector $X = \{x_1, x_2, ..., x_n\}$ such that $\sum_{i=1}^{n} x_i w_i \leq B$, and $\sum_{i=1}^{n} x_i v_i \geq K$. Given such an instance, we construct a star network using the following transformation: $W = n$, $C = max_i(w_i + v_i) + 1$, $Q = K + \sum_{i=1}^{n}(C - w_i - v_i)$ and the traffic matrix is represented on the Figure 1. In this figure the nodes are the nodes of the star network (the hub is not represented), and the links represent the traffic demands. Traffic demands equal to 0 are not represented, and the value on the link from a node a to a node b is $t_{a,b}$. Nodes from $n+1$ to $n+10$ represent each one a node of the network, but nodes $i_\alpha, j_\alpha, k_\alpha, l_\alpha, m_\alpha, p_\alpha$ and q_α represent each one several nodes:
For the nodes i_α, α ranges from 1 to n (i.e. i_α represents the nodes $i_1, i_2, ..., i_n$);
for the nodes j_α, α ranges from 1 to $\lfloor \frac{(n-2)C}{C-1} \rfloor$;
for the nodes k_α, α ranges from 1 to $\lfloor \frac{\sum_{k=1}^{n}(w_k - B)}{C-1} \rfloor$;

for the nodes l_α, α ranges from 1 to $\lfloor \frac{nC-(C-1)}{C-1} \rfloor$;
for the nodes m_α, α ranges from 1 to $\lfloor \frac{nC-((\sum_{k=1}^{n}(w_k-B)) \bmod (C-1))}{C-1} \rfloor$;
for the nodes p_α, α ranges from 1 to $\lfloor \frac{n}{C-1} \rfloor$;
for the nodes q_α, α ranges from 1 to $\lfloor \frac{nC-n((n-2)C \bmod (C-1))}{C-1} \rfloor$;
and for the nodes r_α, α ranges from 1 to $\lfloor \frac{nC-\sum_{k=1}^{n} w_k}{C-1} \rfloor$.

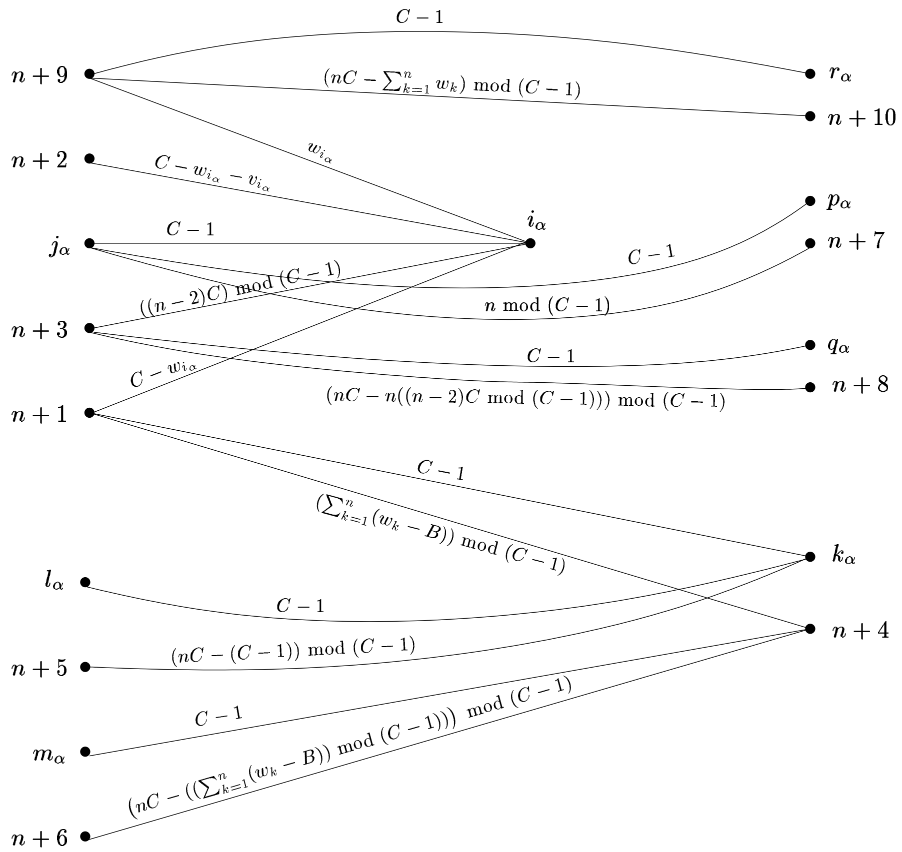

Fig. 1. Illustration of the traffic matrix. Transmitters are on the left and receivers on the right.

Lemma 1 *Let a be a transmitter and b a receiver. It is not possible to have a lightpath from a to b, if $(a,b) \neq (n+1, i_\alpha)$ or $(a,b) \neq (n+2, i_\alpha)$.*

Proof: Let us show that each traffic demand different from 0 between each couple (a,b) of nodes in $V_1 \times V_2$ ($(a,b) \neq (n+1, i_\alpha)$ and $(n+2, i_\alpha)$), cannot be optically routed. In order to show that it is not possible to route $t_{a,b}$ optically, we will see that either the sum of the traffic streams from a, or the sum of the traffic streams to b, is equal to nC, and that $t_{a,b}$ is smaller than C.

- $\forall x \in V_2, t_{n+9,x}$ cannot be optically routed. Indeed:
$$\begin{aligned}\sum_{x\in V_2} t_{n+9,x} &= \sum_\beta t_{n+9,r_\beta} + t_{n+9,n+10} + \sum_\beta t_{n+9,i_\beta} \\ &= \lfloor \frac{nC-\sum_{k=1}^n w_k}{C-1} \rfloor (C-1) + (nC - \sum_{k=1}^n w_k) \bmod (C-1) \\ &\quad + \sum_{k=1}^n w_k \\ &= nC\end{aligned}$$
and $t_{n+9,r_\alpha} < C,\ t_{n+9,n+10} < C,\ t_{n+9,i_\alpha} < C$.
- $\forall x \in V_2, t_{j_\alpha,x}$ cannot be optically routed. Indeed:
$$\begin{aligned}\sum_{x\in V_2} t_{j_\alpha,x} &= \sum_\beta t_{j_\alpha,i_\beta} + \sum_\beta t_{j_\alpha,p_\beta} + t_{j_\alpha,n+7} \\ &= n(C-1) + \lfloor \frac{n}{C-1} \rfloor (C-1) + n \bmod (C-1) \\ &= nC\end{aligned}$$
and $t_{j_\alpha,i_\beta} < C,\ t_{j_\alpha,p_\beta} < C,\ t_{j_\alpha,n+7} < C$.
- $\forall x \in V_2, t_{n+3,x}$ cannot be optically routed. Indeed:
$$\begin{aligned}\sum_{x\in V_2} t_{n+3,x} &= \sum_\beta t_{n+3,i_\beta} + \sum_\beta t_{n+3,q_\beta} + t_{n+3,n+8} \\ &= n((n-2)C \bmod (C-1)) + \lfloor \frac{nC-n((n-2)C \bmod (C-1))}{C-1} \rfloor (C-1) \\ &\quad + (nC - n((n-2)C \bmod (C-1))) \bmod (C-1) \\ &= nC\end{aligned}$$
and $t_{n+3,i_\alpha} < C,\ t_{n+3,q_\alpha} < C,\ t_{n+3,n+8} < C$.
- $\forall x \in V_1, t_{x,k_\alpha}$ cannot be optically routed. Indeed:
$$\begin{aligned}\sum_{x\in V_1} t_{x,k_\alpha} &= t_{n+1,k_\alpha} + \sum_\beta t_{l_\beta,k_\alpha} + t_{n+5,k_\alpha} \\ &= (C-1) + \lfloor \frac{nC-(C-1)}{C-1} \rfloor (C-1) + (nC-(C-1)) \bmod (C-1) \\ &= nC\end{aligned}$$
and $t_{n+1,k_\alpha} < C,\ t_{l_\beta,k_\alpha} < C,\ t_{n+5,k_\alpha} < C$.
- $\forall x \in V_1, t_{x,n+4}$ cannot be optically routed. Indeed:
$$\begin{aligned}\sum_{x\in V_1} t_{x,n+4} &= \sum_\beta t_{m_\beta,n+4} + t_{n+6,n+4} + t_{n+1,n+4} \\ &= \frac{nC-((\sum_{k=1}^n (w_k-B)) \bmod (C-1))}{C-1}(C-1) + (\lfloor nC - ((\sum_{k=1}^n (w_k- \\ &\quad B)) \bmod (C-1)) \rfloor) \bmod (C-1) + (\sum_{k=1}^n (w_k-B)) \bmod (C-1) \\ &= nC\end{aligned}$$
and $t_{n+1,n+4} < C,\ t_{m_\beta,n+4} < C,\ t_{n+6,n+4} < C$. □

Lemma 2 *Let $\alpha \in \{1,...,n\}$. Traffic demands t_{n+1,i_α} and t_{n+2,i_α} cannot be optically routed simultaneously.*

Proof: The node i_α receives from the hub a total traffic equal to: $t_{n+9,i_\alpha} + \sum_\beta t_{j_\beta,i_\alpha} + t_{n+3,i_\alpha} = w_{i_\alpha} + \lfloor \frac{(n-2)C}{C-1} \rfloor (C-1) + ((n-2)C) \bmod (C-1) = (n-2)C + w_{i_\alpha} > (n-2)C$
Since $W = n$, there is at most one wavelength left to have a lightpath to the node i_α. □

Lemma 3 *Let $\alpha \in \{1,...,n\}$. It is possible to have a lightpath from $n+1$ to i_α, or from $n+2$ to i_α.*

Proof:

- Let us show that it is possible to have a lightpath from $n+1$ to i_α: $\sum_{x \in V_1, x \neq n+1} t_{x,i_\alpha} = (n-2)C + w_{i_\alpha} + (C - w_{i_\alpha} - v_{i_\alpha}) = (n-1)C - v_{i_\alpha} \leq (n-1)C$
and, since $\exists i_\alpha \in \{1,...,n\}, B \geq w_{i_\alpha}$ (otherwise the instance of the Knapsack problem would be trivial) and $C = max_{i_\alpha}(v_{i_\alpha} + w_{i_\alpha}) + 1$, we have $\sum_{x \in V_2, x \neq i_\alpha} t_{n+1,x} = \sum_{k=1}^{n}(w_k - B) \leq (n-1)C$. So there is enough bandwidth to have a lightpath from $n+1$ to i_α.
- Let us show that it is possible to have a lightpath from $n+2$ to i_α: $\sum_{x \in V_1, x \neq n+2} t_{x,i_\alpha} = (n-2)C + w_{i_\alpha} + (C - w_{i_\alpha}) = (n-1)C \leq (n-1)C$ and $\sum_{x \in V_2, x \neq i_\alpha} t_{n+2,x} = 0 \leq (n-1)C$. So there is enough bandwidth to have a lightpath from $n+2$ to i_α. □

Since t_{n+1,i_α} and t_{n+2,i_α} ($\alpha \in \{1,...,n\}$) are the only traffic demands which can be optically routed (Lemma 1) and since, for each $\alpha \in \{1,...,n\}$ it is possible to have a lightpath from $n+1$ to i_α, or from $n+2$ to i_α, (Lemma 3) but not both (Lemma 2), we need only to consider solutions in which there is a lightpath from exactly one of the nodes $n+1$, $n+2$, to each node i_α to determine the satisfiability of the instance.

Let X denote a candidate solution of the Knapsack instance. Consider the solution of the grooming problem in which X (respectively, $\overline{X}$) represents the indicator vector of the lightpaths formed from node $n+1$ (resp., $n+2$). Nodes i_α are numbered from 1 to n: let $\alpha \in \{1,...,n\}$, we have $i_\alpha = \alpha$. Applying the transformation to the satisfiability criteria of Knapsack, we obtain:

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} x_i w_i \leq B \\
\Leftrightarrow &\textstyle\sum_{i=1}^{n} x_i(C - t_{n+1,i}) \leq \sum_{i=1}^{n}(C - t_{n+1,i}) - (t_{n+1,n+4} + \sum_\beta t_{n+1,k_\beta}) \\
\Leftrightarrow &\textstyle\sum_{i=1}^{n}(\overline{x_i} t_{n+1,i}) + (t_{n+1,n+4} + \sum_\beta t_{n+1,k_\beta}) \leq (n - \sum_{i=1}^{n} x_i)C
\end{aligned}$$

This inequality means that the amount of electronically routed traffic demands (the left hand side of the inequality) has to be smaller than or equal to the capacity of a link, C, times the number of links available (i.e. n minus the number of traffic demands which are electronically routed).

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n} x_i v_i \geq K \\
\Leftrightarrow &\textstyle\sum_{i=1}^{n} x_i(t_{n+1,i} - t_{n+2,i}) \geq Q - \sum_{i=1}^{n} t_{n+2,i} \\
\Leftrightarrow &\textstyle\sum_{i=1}^{n}(x_i t_{n+1,i} + \overline{x_i} t_{n+2,i}) \geq Q
\end{aligned}$$

This inequality means that the total amount of optically routed traffic has to be greater than or equal to Q.

Therefore, a vector X either satisfies both the Knapsack and the grooming instance, or fails to satisfy both. Hence, the grooming instance is satisfiable if and only if the Knapsack instance is.

Theorem 1 *The decision versions of the minimization and the maximization versions of the grooming problem in a passive star are NP-Complete.*

Proof: We already proved that the decision version of the minimization version of the grooming problem in a passive star is NP-Complete. Since we can easily switch from a version to the other one ($OPT_{max} = (\sum_{i \in V_1, j \in V_2} t_{ij}) - OPT_{min}$), the decision version of the maximization version of the grooming problem in a passive star is also NP-Complete. □

3 Polynomial time algorithm for W=2

Let us show that the grooming problem in a passive star is polynomially solvable when the number of wavelengths on each fiber, W, is equal to 2. We will give a polynomial time algorithm which gives an optimal solution of this problem.

First of all, let us remark that in each row (or column) of T where there is at least three values different from 0, we can at most route one traffic demand optically, because each lightpath which is not electronically switched at the hub needs a wavelength for him only, and we have only two wavelengths per fiber. On the contrary, when there is in a row (or column) only two values different from 0, it may be possible to route optically both. We transform the matrix T in a matrix T' in which it is possible to route optically at most one traffic demand for each row, and one traffic demand for each column: if a row of T has two and only two values t_{ij} and $t_{ij'}$ different from 0, we create two rows $i1$ and $i2$ in T' such that $t'_{i1,j} = t_{ij}$, $t'_{i2,j} = t_{ij'}$ and the other values in these rows are 0. Similarly, if a column of T has two and only two values t_{ij} and $t_{i'j}$ different from 0, we create two columns $j1$ and $j2$ in T' such that $t'_{i,j1} = t_{ij}$, $t'_{i,j2} = t_{i'j}$ and the other values in these columns are 0. In this way, there is at most one request per row and one request per column which can be optically routed.

Let us now transform our traffic matrix T' in another matrix $M = [m_{ij}]$, in which we will look for a maximum weighted matching. In order to do that, we will apply the following rule: if a traffic demand t'_{ij} cannot be optically routed (i.e. $\sum_{k \neq j} t'_{ik} > C$ or $\sum_{k \neq i} t'_{kj} > C$) then $m_{ij} = -\infty$. Otherwise, $m_{ij} = t'_{ij}$. Since there is in M at most one request per row and one request per column which can be optically routed, and since a traffic demand m_{ij} is different from $-\infty$ if and only if it is allowed to be optically routed, the result of a maximum weighted matching in the bipartite graph whose adjacent matrix is M, is an optimal result for the traffic grooming problem whose traffic matrix is T.

Theorem 2 *The minimization and the maximization versions of the traffic grooming problem in a passive star are polynomially solvable if the number of wavelengths per fiber is equal to two.*

Proof: The optimal solution of the minimization version and the optimal solution of the maximization version are the same, and the algorithm above solves polynomially these problems. □

4 Approximation algorithms

Theorem 3 *It is not possible to deduce a constant approximation guarantee for the maximization (resp. minimization) version of the traffic grooming problem in a passive star network from a constant approximation algorithm for the minimization (resp. maximization) version.*

Proof: Let OPT_{max} be the cost of an optimal solution of the maximization version (i.e. the maximum amount of traffic which can be optically routed) and OPT_{min} the cost of an optimal solution of the minimization version (i.e. the minimum amount of traffic which is electronically switched at the hub in a feasible solution). Let S be a solution of the traffic grooming problem in a passive star. Let denote $c_{max}(S)$ the cost of the maximization version (i.e. $c_{max}(S)$ is the amount of traffic which is optically routed in the solution S) and $c_{min}(S)$ the cost of the minimization version (i.e. $c_{min}(S)$ is the amount of traffic which is electronically switched at the hub in the solution S).

Let ε_1 and ε_2 be two real numbers such that $0 \leq \varepsilon_1 < 1$ and $0 \leq \varepsilon_2$. Let us show that it is not possible to deduce a $(1-\varepsilon_1)$-approximation guarantee for the maximization version from a $(1+\varepsilon_2)$-approximation algorithm for the minimization version: Consider an instance I of the problem, where we can at most route only one traffic stream optically ($OPT_{max} = 1$ and $OPT_{min} = (\sum_{i\in V_1, j\in V_2} t_{ij}) - 1$). Consider a solution S of I where all the traffic demands are electronically switched at the hub ($c_{max}(S) = 0$ and $c_{min}(S) = \sum_{i\in V_1, j\in V_2} t_{ij}$). If $\sum_{i\in V_1, j\in V_2} t_{ij}$ is large enough, we have $\frac{c_{min}(S)}{OPT_{min}} = \frac{\sum_{i\in V_1, j\in V_2} t_{ij}}{(\sum_{i\in V_1, j\in V_2} t_{ij})-1} \leq 1+\varepsilon_2$ but there is no $\varepsilon_1 < 1$ such that $\frac{c_{max}(S)}{OPT_{max}} = \frac{0}{1}$ is greater than or equal to $1-\varepsilon_1$.

Let us show that it is possible to have an instance such that $\sum_{i\in V_1, j\in V_2} t_{ij}$ is as big as we wish and where we can at most route one traffic stream optically: consider the instance where we have $2W$ transmitters, 2 receivers, and the traffic matrix $[t_{ij}]$ is such that $t_{1,1} = 1$; $\forall i \in \{2,3,...,2W\}$, $t_{i,1} = 0$; and $\forall i \in \{1,...,2W\}$, $t_{i,2} = \frac{C}{2}$. The only traffic demand which can be optically routed is $t_{1,1}$ because the second receiver receives WC traffic streams, and each traffic demand is different from C.

Similarly, let us show that it is not possible to deduce a $(1+\varepsilon_2)$-approximation guarantee for the minimization version from a $(1-\varepsilon_1)$-approximation algorithm for the maximization version: Consider an instance I of the problem, where all the traffic demands can be optically routed ($OPT_{max} = \sum_{i\in V_1, j\in V_2} t_{ij}$ and $OPT_{min} = 0$). It is trivial that such an instance exists. Consider a solution S of I where all the traffic streams, except two, are optically routed ($c_{max}(S) = (\sum_{i\in V_1, j\in V_2} t_{ij}) - 2$ and $c_{min}(S) = 2$). If $\sum_{i\in V_1, j\in V_2} t_{ij}$ is large enough, we have $\frac{c_{max}(S)}{OPT_{max}} = \frac{(\sum_{i\in V_1, j\in V_2} t_{ij})-2}{\sum_{i\in V_1, j\in V_2} t_{ij}} \geq 1-\varepsilon_1$ but there is no $\varepsilon_2 \geq 0$

such that $\frac{c_{min}(S)}{OPT_{min}} = \frac{2}{0}$ is smaller than or equal to $1 + \varepsilon_2$. □

4.1 Approximation algorithm for the minimization version

Let us give an integer programming formulation of the minimization version of the traffic grooming problem in a passive star. Let denote $x_{ij} \in \{0,1\}$ a variable which indicates whether the traffic demand t_{ij} is optically routed ($x_{ij} = 0$) or electronically switched at the hub ($x_{ij} = 1$). The objective is to:

$$\text{Minimize} \sum_{i \in V_1, j \in V_2} x_{ij} t_{ij} \tag{1}$$

We have two types of constraints:
constraints on the frequencies:

$$\forall i \in V_1, \sum_{j \in V_2} x_{ij} \geq |V_2| - W \tag{2}$$

$$\forall j \in V_2, \sum_{i \in V_1} x_{ij} \geq |V_1| - W \tag{3}$$

constraints on the traffic:

$$\forall i \in V_1, \sum_{j \in V_2} (1 - x_{ij})(C - t_{ij}) \leq WC - \sum_{j \in V_2} t_{ij} \tag{4}$$

$$\forall j \in V_2, \sum_{i \in V_1} (1 - x_{ij})(C - t_{ij}) \leq WC - \sum_{i \in V_1} t_{ij} \tag{5}$$

Inequalities (2) (resp.(3)) mean that at most W traffic demands per transmitter (resp. receiver) can be optically routed, because we need one wavelength for each traffic demand optically routed. Inequalities (4) and (5) mean that the unused space $(C - t_{ij})$ left when t_{ij} is optically routed, has to be smaller than the free space available (i.e. the total amount of bandwidth minus the total amount of traffic demands). It is also equivalent to say that the amount of electronically routed traffic demands has to be smaller than the capacity of a link, C, times the number of links available (i.e. W minus the number of traffic demands which are optically routed).
Constrains (4) and (5) are equivalent to:

$$\forall i \in V_1, \sum_{j \in V_2} x_{ij}(C - t_{ij}) \geq C(|V_2| - W) \tag{6}$$

$$\forall j \in V_2, \sum_{i \in V_1} x_{ij}(C - t_{ij}) \geq C(|V_1| - W) \tag{7}$$

Note that the constraints on the traffic imply the constraints on the frequencies: if we divide by C the constraints on the frequencies we have:

$$\forall i \in V_1, \sum_{j \in V_2} x_{ij}(1 - \frac{t_{ij}}{C}) \geq |V_2| - W \tag{8}$$

$$\forall j \in V_2, \sum_{i \in V_1} x_{ij}(1 - \frac{t_{ij}}{C}) \geq |V_1| - W \tag{9}$$

Since $t_{ij} \leq C$, these last inequalities imply the constraints on the frequencies (2) and (3).

So we can formulate our problem in the following way:

$$\text{Minimize} \sum_{i \in V_1, j \in V_2} x_{ij} t_{ij} \tag{10}$$

subject to:

$$\forall i \in V_1, \sum_{j \in V_2} x_{ij}(C - t_{ij}) \geq C(|V_2| - W) \tag{11}$$

$$\forall j \in V_2, \sum_{i \in V_1} x_{ij}(C - t_{ij}) \geq C(|V_1| - W) \tag{12}$$

$$\forall i \in V_1, j \in V_2, x_{ij} \in \{0, 1\} \tag{13}$$

Theorem 4 *There exists a polynomial time H_{2C}-approximation algorithm for the minimization version of the traffic grooming problem in a passive star.*

Proof: We will transform the minimization version of the traffic grooming problem in a passive star into a constrained multiset multicover problem. Given a universal set $\mathcal{U}$, a collection of subsets of $\mathcal{U}$, $T = \{S_1, S_2, ..., S_k\}$, and a cost function $c : T \to Q^+$, the *set cover* problem asks for a minimum cost sub-collection $\mathcal{C} \in T$ that covers all the elements of $\mathcal{U}$ (i.e. $\bigcup_{S \in \mathcal{C}} S = \mathcal{U}$). The *multiset multicover* is a natural generalization of the set cover problem: in this problem each element e occurs in a multiset S with arbitrary multiplicity denoted $m(S, e)$, and each element e has an integer coverage requirement r_e, which specifies how many times e has to be covered. In the *constrained multiset multicover* problem, each subset $S \in T$ is chosen at most once. Thus the integer program is: Minimize $\sum_S c(S) x_S$ subject to $\sum m(S, e) x_S \geq r_e$ and $x_S \in \{0, 1\}$.

Let us show the transformation of the traffic grooming problem into the constrained multiset multicover problem. This transformation comes directly from the integer programming formulation of the problem: for each request from the transmitter $i \in V_1$, denoted by e_i, to the receiver $j \in V_2$, denoted by r_j, we create the subset S_{ij} which contains $(C - t_{ij})$ times e_i and $(C - t_{ij})$ times r_j. The cost of this subset is $c(S_{ij}) = t_{ij}$. The covering requirement of each element $e \in V_1$ is $r_e = C(|V_2| - W)$, and the covering requirement of each element $e \in V_2$

is $r_e = C(|V_1| - W)$.

S. Rajagopalan and V. Vazirani give in [3] a greedy approximation algorithm for the constrained multiset multicover problem. This algorithm consists in iteratively picking the most cost-effective set from T and removing this set from T. The cost-effectiveness of a set S is the average cost at which it covers new elements, i.e. the cost of S divided by the number of its elements which are not yet covered. They proved that this algorithm has an approximation guarantee of H_k, the k-th harmonic number (i.e. $H_k = 1 + \frac{1}{2} + ... + \frac{1}{k}$), where k is the size of the largest multiset in the given instance. In our case, the size of a multiset S_{ij} is $2C - 2t_{ij}$, which is smaller than $2C$. So we obtain a solution of the traffic grooming problem which has an approximation guarantee of $H_{2C} \leq log(2C) + 1$. □

4.2 Approximation algorithm for the maximization version

The maximization version of the traffic grooming problem in a passive star is the following one:

$$\text{Maximize} \sum_{i \in V_1, j \in V_2} y_{ij} t_{ij} \tag{14}$$

subject to:

$$\forall i \in V_1, \sum_{j \in V_2} y_{ij}(C - t_{ij}) \leq CW - \sum_{j \in V_2} t_{ij} \tag{15}$$

$$\forall j \in V_2, \sum_{i \in V_1} y_{ij}(C - t_{ij}) \leq CW - \sum_{i \in V_1} t_{ij} \tag{16}$$

$$\forall i \in V_1, j \in V_2, y_{ij} \in \{0, 1\} \tag{17}$$

Here y_{ij} indicates whether t_{ij} is optically routed ($y_{ij} = 1$) or electronically switched at the hub ($y_{ij} = 0$). This integer programming formulation of the problem is obtained by replacing x_{ij} in the integer programming formulation of the minimization version by $1 - y_{ij}$.

Theorem 5 *There exists a polynomial time* $(2 + \frac{4}{5})$*-approximation algorithm for the maximization version of the traffic grooming problem in a passive star.*

Proof: Let us now transform this problem into a *demand matching problem* [4]. The demand matching problem is the following one: take a graph $G = (V, E)$ and let each node $v \in V$ have an integral *capacity*, denoted by b_v. Let each edge $e = (u, v) \in E$ have an integral *demand*, denoted by d_e. In addition, associated with each edge $e \in E$ is a *profit*, denoted by p_e. A *demand matching* is a subset $M \subseteq E$ such that $\sum_{e \in \delta(v) \cap M} d_e \leq b_v$ for each node v. Here $\delta(v)$ denotes the set of edges of G incident to v. The demand matching problem is to find a demand matching which maximizes $\sum_{e\in M} p_e$. Thus the integer program is: Maximize $\sum_{e\in E} \frac{p_e}{d_e} x_e$ subject to: $\forall v \in V, \sum_{e\in\delta(v)} x_e \le b_v$ and $\forall e \in E, x_e \in \{0, d_e\}$.

B. Shepherd and A. Vetta showed in [4] that a randomized algorithm provides a factor $(2+\frac{4}{5})$-approximation guarantee for the demand matching problem in bipartite graphs.

Let us show the transformation of the traffic grooming problem in a passive star into the demand matching problem: $\forall e \in (V_1 \times V_2),\ x_e = y_e(C - t_e)\ ,\ p_e = t_e\ ,\ d_e = C - t_e$. $\forall v \in V_1,\ b_v = CW - \sum_{j\in V_2} t_{vj}$ and $\forall v \in V_2,\ b_v = CW - \sum_{i\in V_1} t_{iv}$.

Since there is a factor $(2+\frac{4}{5})$-approximation algorithm for the demand matching problem, there is also a factor $(2+\frac{4}{5})$-approximation algorithm for the traffic grooming problem in a passive star. □

Conclusion

We showed in this paper that the traffic grooming problem in a passive star is NP-Complete, in both the minimization and the maximization versions of the problem. We showed that these problems are polynomially solvable if there are two wavelengths per fiber: we gave an algorithm which gives an optimal solution. We showed that we cannot deduce a constant approximation guarantee of the maximization (resp. minimization) version from a constant approximation guarantee of the minimization (resp. maximization) version of the problem. We gave two approximation algorithms and we obtained an approximation guarantee of H_{2C} for the minimization version and an approximation guarantee of $(2+\frac{4}{5})$ for the maximization version. Since the solutions returned by these algorithms are all solutions of the maximization version as well as solutions of the minimization version of the problem, it would be interesting to program these algorithms and compare their results.

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