Gradual Generalized Modus Ponens

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Abstract—Gradual relationship between premises and conclusions is often an underlying property of fuzzy rules. In this paper, we propose to integrate the gradual hypothesis, sometimes called monotonicity, to Generalized Modus Ponens (GMP).

To achieve this objective, we defined the Gradual Generalized Modus Ponens (GGMP), based on a partitioning of the universe of discourse. Moreover, we prove, for this formulation, some major preservation properties, such as for the convexity, the continuity and the normality. Finally, we show that the ordering of different fuzzy observations is conserved for the associated conclusions.

Keywords—Generalized Modus Ponens, Gradual Rules, Graduality, Approximate Reasoning, Monotonicity

I. INTRODUCTION

Since its introduction, in L. A. Zadeh’s paper [1], Generalized Modus Ponens (GMP) has become one of the most powerful tools in approximate reasoning. However, GMP has been used without any assumptions, which if verified, would increase the specificity of the inferred conclusion. One such hypothesis is the gradual relationship between the premise and the consequence. For instance for a washing rule: “if clothes weigh around 8 kilograms then use more or less 10 liters of water,” a gradual hypothesis can be defined: “the more weight, the more water”. Note that this is not the “graduality” meant by some authors, who focus on the truth-values: the more it is 8kg the more it should be 10l.

In the literature, there are several research works on graduality in a sense close to the one presented in this work. They appear under what we can call studies on the monotonicity of fuzzy inference systems (FIS). In [2] and [3], Broekhoven et al. show that in a Mamdani-Assilian model, ordered linguistic values for all input variables and for all output variables, plus a set of rules describing a monotone system are not enough to guarantee a monotone input-output behavior. They state that the choice of the mathematical operators used when calculating the model output and the properties of the membership functions are also of crucial importance. These constraints have been observed as well as in another work by M. Štěpnička et al. [4].

In [5] and [6], K. M. Tay et al. propose an approach to build a fuzzy inference system (FIS) that preserves the monotonicity property. The authors introduce a fuzzy re-labeling technique to re-order the consequences of fuzzy rules in the database and a monotonicity index. This approach is able to overcome several restrictions; in particular it uses a similarity-based reasoning scheme to design monotonic multi-input FIS models.

There are also some studies ([7], [8]) on the monotonicity of single input rule modules (SIRM)s where the output of the rule is generally simple, i.e., a precise value. The authors study the properties of the inference method and especially the conditions for the monotonicity of the inference results.

All the above studies discuss how to guarantee the monotonicity of the output of the whole inference system. In this paper, we propose a modification of GMP in order to integrate the graduality. We call this new inference Gradual GMP (GGMP). More precisely, the proposed approach introduces monotonicity of the output, with respect to the input, for each individual rule, which is known by the user to verify a graduality hypothesis.

In the next section, we introduce our notation by briefly recalling the fundamentals of GMP. GGMP approach is defined in section III. Before concluding in section V, we present in section IV important properties of GGMP, in particular those concerning order conservation.

II. GENERALIZED MODUS PONENS

Generalized Modus Ponens is a key inferring mechanism in approximate reasoning. Let X and Y be two variables in the universes of discourse U and V. A and A' are fuzzy subsets of U, and B is a fuzzy subset of V. The most general form of GMP is given below:

\[
\text{If } X \text{ is } A \text{ then } Y \text{ is } B
\]

where the membership function of B' is defined by:

\[
\forall y \in V, \mu_{B'}(y) = \sup_{x \in U} T(\mu_A(x), \mathcal{I}(\mu_A(x), \mu_B(y)))
\]

with \(T\) is a triangular norm, and \(\mathcal{I}\) is a fuzzy implication operator. Table I lists some implications and triangular norms which are compatible with normal GMP.

From the definition, obviously GMP does not include a gradual hypothesis, because there is no relationship between the different x or y values. In particular, all the universe of the premise is considered in the inferring for each point of the conclusion. Therefore, if there exists a graduality relationship between the premise and the consequence, GMP does not exploit it (Fig. 1).

III. GRADUAL GENERALIZED MODUS PONENS

As mentioned above, very often, underlying a rule, there is an unexploited graduality hypothesis. For example, in the rule “if clothes weigh around 8 kilograms then use more or less 10
defined as follows: i.e. the interval so that

In this study, the membership functions are assumed to
be convex, normalized and continuous, and their supports are
required to be bounded. In this sense, fully continuous fuzzy
numbers and fuzzy intervals can be used with the proposed
required to be bounded. In this sense, fully continuous fuzzy
functions computed on the basis of the

We define the three parts Smaller, Greater and Indistinguishable of fuzzy set $A$, using three membership functions computed on the basis of the kernel and complement of $A$ (Fig. 2).

Let us refer to the kernel of a fuzzy set $A$ by $[A_L, A_R]$, i.e. the interval so that $\forall x \in [A_L, A_R] \ \mu_A(x) = 1$.

The membership function of the Smaller part of $A$ is defined as follows:

$$\varphi_{\text{Smaller}_A}(x) = \begin{cases} \mu_A(x), & x < A_L \\ 1 - \mu_A(x), & A_L \leq x \leq A_R \\ 0, & x > A_R \end{cases}$$

(b) Observation on the left

Fig. 1. Normal Generalized Modus Ponens

The membership function of the Greater part of $A$ is:

$$\varphi_{\text{Greater}_A}(x) = \begin{cases} \mu_A(x), & x > A_R \\ 1 - \mu_A(x), & x \leq A_R \end{cases}$$

The membership function of the Indistinguishable part of $A$ is:

$$\varphi_{\text{Indistinguishable}_A}(x) = \begin{cases} 0, & x \notin [A_L, A_R] \\ 1, & x \in [A_L, A_R] \end{cases}$$

Formal definition of the GGMP

In order to induce a gradual behavior in the GMP, we compute the conclusion membership function $B'$ as follows:

$$\forall y \in V, \mu_{B'}(y) = \sup_{x \in \psi(y)} \mathcal{T}(\mu_A(x), \mathcal{I}(\mu_A(x), \mu_B(y)))$$
where
\[
\psi(y) = \{ x \in U | \varphi_P(x) = \varphi_P(y), \text{ and } \varphi_P(y) > 0, \\
P \in \{ \text{Small, Greater, Indistinguishable} \} \}
\]  

(6)

Fig. 3 shows the conclusion \( B' \) inferred using GGMP, that should be compared with what would have been obtained using GMP, as shown on Fig. 1.

Inverse graduality

So far, we have just discussed the graduality of the form “the more, the more” or “the less, the less”. There may be rule bases which have inverse gradual hypothesis, i.e., of the form “the more, the less” or “the less, the more”. In this case, we still could use previous definition of GGMP with a slight modification. If we are inferring the degree at \( y \), which is for instance in the Smaller part of \( B \), then \( x \) in the Greater part of \( A \) should be used; similarly, if \( y \) is in the Greater part of \( B \), \( x \) should be in the Smaller part of \( A \). And there is no change if \( y \) is in the Indistinguishable part of \( B \), i.e., \( x \) should be considered in the Indistinguishable part of \( A \).

IV. Properties

A. Compatibility

The compatibility between a t-norm and implication for the Generalized Modus Ponens [9] translates the requirement that if the observation is identical with the premise \((A' \equiv A)\), the inferred conclusion should also be identical with the consequence of the rule \((B' \equiv B)\).

Property 1: Implications \( I_{RG}, I_{BG}, I_G, I_L \) with t-norms \( T_L, T_M \) and \( T_P \) are compatible for the GGMP, in a sense analogous to the GMP (i.e. when \( A' \equiv A \) then \( B' \equiv B \)).

Proof: When \( A' \) is identical with \( A \), then for each \( y \in V \):
\[
\mu_{B'}(y) = \sup_{{x \in \psi(y)}} T(\mu_A(x), I(\mu_A(x), \mu_B(y)))
\]

In case of implications \( I_{RG}, I_{BG}, I_G, I_L \), we have
\[
\mu_A(x) = \mu_B(y) \quad (\text{since } 1 - \mu_A(x) = 1 - \mu_B(y))
\]

thus \( I(\mu_A(x), \mu_B(y))) = 1 \), therefore
\[
\mu_{B'}(y) = \sup_{{x \in \psi(y)}} T(\mu_B(y), 1) = \mu_B(y)
\]

From this point, for short, when talking about GGMP, the implications and t-norms used are assumed to be compatible.

B. Convexity

Convexity is an important property of a fuzzy set, and more generally for approximate reasoning and fuzzy control. For an inference, and in particular for a gradual inference as defined here, it is crucial that convex conclusions are guaranteed in presence of convex observations and rule elements.
Intuitively, convexity means its membership function has only one peak. The following lemma shows formally that a convex fuzzy set has only one peak, i.e., it has no more than one upward side and no more than one downward side.

Because of the membership restrictions assumed in this paper, there exists a maximal value of the membership degree of the fuzzy sets:

\[ \sup_{x \in U} \mu_A(x) = \max_{x \in U} \mu_A(x) = 1 \]

**Lemma 1:** Let \( A \) be a fuzzy set and \( x_0 \in U \) be a point such that \( \mu_A(x_0) = \max_{x \in U} \mu_A(x) \). \( A \) is convex if and only if \( \mu_A(x) \) is non-decreasing in \( (-\infty, x_0] \) and non-increasing in \( [x_0, \infty) \).

**Proof:** For \( A \) convex, we will show that \( \mu_A(x) \) is non-decreasing in \( (-\infty, x_0] \) and non-increasing in \( [x_0, \infty) \). If \( A \) is convex. Using L. Zadeh’s [10] definition to prove that \( A \) is convex. For all \( x_1, x_2 \in U \), without loss of generality, we assume that \( x_1 \leq x_2 \). There are three cases:

i) If \( x_1 \leq x_2 \leq x_0 \), then \( \forall \lambda \in [0, 1] \):

\[ \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \]

ii) If \( x_0 \leq x_1 \leq x_2 \), then \( \forall \lambda \in [0, 1] \):

\[ \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \]

iii) If \( x_1 \leq x_0 \leq x_2 \), we have two cases:

- if \( x_1 \leq \lambda x_1 + (1 - \lambda)x_2 \leq x_0 \) then

\[ \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \]

- if \( x_0 \leq \lambda x_1 + (1 - \lambda)x_2 \leq x_2 \) then

\[ \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \]

Which means \( A \) is convex.

**Property 3:** If \( A, A' \) and \( B \) are convex, the conclusion of GGMP is convex.

**Proof:** Let \( x_0 \) be a point in the kernel of \( A' \), i.e., \( \mu_A(x_0) = 1 \), then \( \varphi_P(y_0) = \varphi_P(x_0) > 0 \), \( P \in \{ \text{Smaller}, \text{Indistinguishable}, \text{Greater} \} \).

First, we will show that \( \mu_B(y) \) is non-decreasing in \( (-\infty, y_0] \), i.e., \( \mu_B(y) \leq \mu_B(y_0) \), \( \forall y \leq y_0 \). For each \( y \leq y_0 \), by using GGMP, we determine \( x^* \) such that:

\[ x^* = \arg \max_{x \in \psi(y)} T(\mu_A(x), \mathcal{I}(\mu_A(x), \mu_B(y))) \]

From the definition of \( x_0, y_0 \), and \( x^* \), we have:

\[ \mu_A(x_0) = \mu_B(y_0) \]

\[ \mu_A(x^*) = \mu_B(y) \]

(because \( x_0 \in \psi(y_0) \), and \( x^* \in \psi(y) \)).

There are four cases of the positions of \( y \) and \( y_0 \):

i) If \( y \) and \( y_0 \) are in the **Indistinguishable** part of \( B \), i.e.,

\[ \mu_B(y) = \mu_B(y_0) = 1 \]

then \( \mu_A(x^*) = \mu_A(x_0) = 1 \), therefore \( \mu_B(y) = \mu_B(y_0) \).

ii) If \( y \) and \( y_0 \) are in the **Greater** part of \( B \),

\[ y \leq y_0 \implies \mu_B(y) \geq \mu_B(y_0) \]

Therefore \( \mu_A(x^*) \geq \mu_A(x_0) \), thus \( x^* \leq x_0 \) (since \( \mu_A(x) \) is non-increasing in the **Greater** part of \( A \)).

This implies \( \mu_A(x^*) \leq \mu_A(x_0) \) (left side of the kernel of \( A' \)).

Then we have \( \mu_B(y) \leq \mu_B(y_0) \) (monotonicity property of \( t \)-norm).

iii) If \( y \) and \( y_0 \) are in the **Smaller** part of \( B \), i.e.,

\[ y \leq y_0 \implies \mu_B(y) \leq \mu_B(y_0) \]

Thus \( \mu_A(x^*) \leq \mu_A(x_0) \), therefore \( x^* \leq x_0 \) (**Smaller** part of \( A \)), thus \( \mu_A(x^*) \leq \mu_A(x_0) \).

Then we have \( \mu_B(y) \leq \mu_B(y_0) \).

iv) If \( y \) is in the **Smaller** part and \( y_0 \) is in the **Indistinguishable** part of \( B \); or \( y \) is in the **Indistinguishable** part and \( y_0 \) is in the **Greater** part of \( B \); or \( y \) is in the **Smaller** part and \( y_0 \) is in the **Greater** part of \( B \), then respectively, in these cases, we have obviously:

\[ x^* \leq x_0 \implies \mu_A(x^*) \leq \mu_A(x_0) \]

\[ \mu_B(y) \leq \mu_B(y_0) \]

From the previous cases, we conclude that \( \mu_B(y) \) is non-decreasing in \( (-\infty, y_0] \).

Similarly, we could prove that \( \mu_B(y) \) is non-increasing in \( [y_0, \infty) \), i.e., \( B' \) is convex.

**C. Normality**

Fuzzy set \( A \) is said to be normal if the greatest value of the membership function of \( A \) is unity, i.e., \( \sup_{x \in U} \mu_A(x) = 1 \).

Sometimes, in particular conditions, the fuzzy sets are not normal. In general, the guarantee that from normal sets we
obtain a normal fuzzy set is a nice property in the context of reasoning mechanisms.

Property 4: If \( A, A' \) and \( B \) are normal, the conclusion of GGMP is normal, when using implications \( I_{RG}, I_{BG}, I_G, I_L \) with compatible t-norms \( T_L, T_M \) and \( T_P \).

Proof: Let \( x_0 \) be a point in the kernel of \( A' \), and let \( y_0 \) be a point such that
\[
\varphi_{P_0}(y_0) = \varphi_{P_A}(x_0) > 0
\]
for some \( P \) in \{Smaller, Greater, Indistinguishable\}.

From the definition of \( y_0 \), we have \( x_0 \in \psi(y_0) \), and
\[
T(\mu_A(x_0), \mu_A(x_0), \mu_B(y_0))
\]
\[
= T(1, 1) = 1
\]
Since \( I(\mu_A(x_0), \mu_B(y_0)) = 1 \) for implications \( I_{RG}, I_{BG}, I_G, I_L \).

Therefore
\[
\mu_B(y_0) = \sup_{x \in \psi(y)} T(\mu_{A'}(x), \mu_A(x), \mu_B(y)))
\]
\[
= T(\mu_A(x_0), \mu_A(x_0), \mu_B(y_0))
\]
\[
= 1
\]
Which means \( B' \) is normal.

Taking into account the membership restrictions, this property just guarantees that any obtained conclusion will be normal. Also, we notice that this property applies for compatible implication and t-norm pairs according to property 1, but not for implication \( I_M \) with its compatible t-norm \( T_M \).

D. Continuity

Property 5: If \( A, A' \) and \( B \) are continuous, the conclusion of GGMP is continuous.

Proof: By the definition of GGMP, for each \( y \), there always exists \( x \) which fulfills the condition of \( \psi(y) \), thus the conclusion is continuous.

E. Ordering

When integrating the gradual hypothesis into GMP, ordering can be seen as a fundamental associated property. For instance, if the observation is smaller than the premise, then we expect the inferred conclusion to be smaller than the consequence of the rule.

However, in fuzzy logic, ordering is not uniquely defined. There are several measures [11], [12] in the literature, going from simple to very complex methods, yet none of them provides what could be called a unique ordering of fuzzy sets.

In this paper, we choose to focus on two ordering methods: the fuzzy max order introduced by J. Ramik and J. Rímanek in [13], and the \( \alpha \)-preference order presented by Adamo in [14]. The former is a partial order relation, expressing the simple but obvious ordering. The latter, more flexible, is an instance of a family of ordering methods, based on a ranking function and was used in applications by many authors [15]. Fig. 4 and Fig. 5 illustrate the conservation of the order for, respectively, the fuzzy max order and the \( \alpha \)-preference order.

Definition 1: (Fuzzy max order) \( A \) is said to be below or equal to \( B \) according to the fuzzy max order \( (A \leq_m B) \) if for all \( \alpha \in (0, 1] \):
\[
\inf\{A_\alpha\} \leq \inf\{B_\alpha\} \quad \text{and} \quad \sup\{A_\alpha\} \leq \sup\{B_\alpha\} \quad (7)
\]

Property 6: Let \( A \) and \( B \) be fuzzy subsets of \( U \) and \( V \); \( A' \) be the observation and \( B' \) the conclusion of GGMP. If \( A' \leq_m A \) then \( B' \leq_m B \); If \( A \leq_m A' \) then \( B \leq_m B' \).

Proof: From the conditions of the membership functions mentioned above, there exist both minimal and maximal elements of the \( \alpha \)-cut of a fuzzy set for all \( \alpha \in (0, 1] \), i.e.,
\[
\inf\{A_\alpha\} \quad \text{and} \quad \sup\{A_\alpha\} = \max\{A_\alpha\}
\]
If \( A \leq_m A' \), for each \( \alpha \in (0, 1] \), we determine \( y_0 \in V \) such that \( y_0 = \inf\{B_\alpha'\} \). There exists \( x_0 \in U \) which is the chosen point to infer \( \mu_B(y_0) \), i.e.,
\[
x_0 = \arg\max\{T(\mu_A(x), \mu_B(y_0))\}
\]
First, we will show that \( \inf\{B_\alpha\} \leq \inf\{B'_\alpha\} \).

i) If \( x_0 \) is in the Indistinguishable or Greater part of \( A \) then \( y_0 \) is also in the corresponding Indistinguishable or Greater part of \( B \), respectively.

Thus we have \( \inf\{B_\alpha\} \leq y_0 = \inf\{B'_\alpha\} \) (since \( \inf\{B_\alpha\} \) belongs to the Smaller part of \( B \)).

ii) If \( x_0 \) is in the Smaller part of \( A \) then \( y_0 \) is in the Smaller part of \( B \).

In case of using implications \( I_{RG}, I_{BG}, I_G, I_L \), we have
\[
\mu_A(x_0) = \mu_B(y_0) = \alpha
\]
then \( \inf\{A_\alpha\} \leq x_0 \)
(from the hypothesis \( \inf\{A_\alpha\} \leq \inf\{A'_\alpha\} \)).

Therefore, \( \mu_B(y_0) = \mu_A(x_0) \geq \alpha \)
(by non-decreasing property of convex fuzzy set in the Smaller part).

This implies \( \inf\{B_\alpha\} \leq y_0 = \inf\{B'_\alpha\} \).

In case of using implication \( I_M \) with t-norm minimum, from the definition of \( x_0 \),
\[
\mu_B(y_0) = \min\{\mu_A(x_0), \mu_A(x_0)\} = \alpha
\]
Here, we have \( \mu_A(x_0) = \alpha \) and \( \mu_A(x_0) \geq \alpha \)
(otherwise, if \( \mu_A(x_0) = \alpha \) and \( \mu_A(x_0) = \alpha_1 > \alpha \),
then \( \inf\{A_{\alpha_1}\} > x_0 \), this is inconsistent with the hypothesis that \( \inf\{A_{\alpha_1}\} \leq \inf\{A'_{\alpha_1}\} \).

Therefore, \( \mu_B(y_0) = \mu_A(x_0) \geq \alpha \),
it implies \( \inf\{B_\alpha\} \leq \inf\{B'_\alpha\} \).

Similarly, we have \( \sup\{B_\alpha\} \leq \sup\{B'_\alpha\} \), i.e., \( B \leq m B' \).

If \( A' \leq_m A \), we prove analogously that \( B' \leq_m B' \).

Remark: With implication \( I_M \), the conclusion of GGMP is not normal, then in the condition of the definitions of these ordering measures, the greatest value of \( \alpha \) is the height of the conclusion, i.e., \( \alpha \in (0, \text{Height}(B')) \).

Definition 2: (\( \alpha \)-preference order) \( A \) is said to be below or equal to \( B \) according to the \( \alpha \)-preference order \( (A \leq_\alpha B) \) if for given \( \alpha \in (0, 1] \):
\[
\sup\{A_\alpha\} \leq \sup\{B_\alpha\} \quad (8)
\]
Property 7: Let $A$ and $B$ be fuzzy subsets of $U$ and $V$; $A'$ be the observation and $B'$ the conclusion of GGMP. For any given $\alpha \in (0, 1]$,

- if $A' \preceq_\alpha A$ then $B' \preceq_\alpha B$;
- if $A \preceq_\alpha A'$ then $B \preceq_\alpha B'$.

Proof: From the definition of the $\alpha$-preference order,

- if $A' \preceq_\alpha A$ then $\sup\{A'_\alpha\} \leq \sup\{A_\alpha\}$

Following steps analogous to those used in the proof of the above theorem, we obtain $\sup\{B'_\alpha\} \leq \sup\{B_\alpha\}$ which means that $B' \preceq_\alpha B$.

Similarly, if $A \preceq_\alpha A'$ then $B \preceq_\alpha B'$.

The order property of the GGMP is even stronger than what is presented by properties 6 and 7, which focus on comparing the observation with the premise and the obtained conclusion with the rule’s consequence. In fact, for GGMP, if one observation is smaller than another observation, then the corresponding conclusions also fulfill this order, i.e., if $A'_1$ is smaller than $A'_2$ then $B'_1$ is smaller than $B'_2$. From the definition of GGMP and the two above properties, we can deduce the following corollary about the ordering of the conclusions according to the ordering of the observations.

Corollary 1: Let $A$ and $B$ be fuzzy subsets of $U$ and $V$; $A'_1$ and $A'_2$ respectively be the conclusions according to the observations $A'_1$ and $A'_2$ using GGMP.

- If $A'_1 \preceq_m A'_2$ then $B'_1 \preceq_m B'_2$
- If $A'_1 \preceq_\alpha A'_2$ then $B'_1 \preceq_\alpha B'_2$
Furthermore, when the observation is entirely in any of the three parts of the premise, GGMP ensures that the derived conclusion is entirely in the equivalent part of the consequence of the rule.

V. Conclusion

In this paper, we propose a formulation of Gradual Generalized Modus Ponens (GGMP), which is able to integrate the gradual hypothesis. To achieve this objective, we use a reduction of the premise search space based on a partitioning of the universe of discourse.

Moreover, we proved that this formulation has several interesting preservation properties, such as the preservation of the convexity, the continuity and the normality. More importantly, we show that the ordering, in the sense of the fuzzy max order and the $\alpha$-preference order, is preserved, not only when comparing the observation with the premise with respect to the inferred conclusion with the rule’s conclusion, but also between several observations with respect to the associated conclusions.

In other words, we show one interesting and viable way of taking into account the gradual hypothesis. One advantage of taking the graduality into account is that it allows a more precise reasoning. Although, from GGMP formulation it is clear that partitioning implies augmented precision. Future works will focus on qualifying and quantifying the precise gain.

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