Similarity-based fuzzy interpolation method

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Abstract

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1 Introduction

Interpolation methods are widely used in many different subjects. In this paper, we refer to the case of a fuzzy rule-based system with sparse rules. We refer to the simplest case, the case with fuzzy rules of the type: (r_i) "If X is A_i, then Y is B_i" where A_i and B_i are fuzzy sets on the universes U and V respectively. Moreover, it is supposed that elements of these universes can be represented by real numbers. With sparse rules we meant that the premises of the rules – the fuzzy sets A_i - do not cover the input space U and thus there exist inputs A such that $A \cap A_i \neq \emptyset$ for all i. In such a case, the computation of the output corresponding to A is the interpolation problem we deal with in this paper. The problem is restricted to compute the output using only the two fuzzy rules (r_1, r_2) such that the premises of the rules are adjacent (no other premise is in between A_1, A_2) and A is in between A_1, A_2. This type of problem is studied firstly in [1] where the method proposed is based on preservation of the proportions. In [5], the authors proved that even though all fuzzy sets (used as premises, conclusions and inputs) are normalized and convex triangular, the solution obtained by Koczy-Hirota method can not be a fuzzy set and they propose some modified method to avoid this problem. Other methods are also proposed in [3], [7], [11] and [2]. In particular, the method presented in [7] is based on the preservation of a measure of similarity between fuzzy sets taking into account the core and the shape of them and in [11] the authors study coherence properties of the extended system obtained adding to the initial system as much rules as poossible inputs. Finally, in [13] a comparative study of the results of different methods proposed are given.

The method proposed in this paper is based on the preservation of conditional consistency and implication measures in the sense of Ruspini (See [12]). The basic ideas of the Ruspini theory are the following:

1. Suppose you have a classical propositional language and a fuzzy similarity relation S on the set of possible interpretations W, i.e. a mapping $S : W \to [0, 1]$ satisfying the conditions of:

   Reflexivity: $S(w, w) = 1$
   Symmetry: $S(w, w') = S(w', w)$
   $\otimes$-transitivity: $S(w, w') \geq S(w, w'') \otimes S(w'', w')$

   where $\otimes$ is a t-norm. Moreover we will impose S to be separating in the sense that $S(w, w') = 1$ if and only if $w = w'$.

2. To each classical proposition p we associate the fuzzy set $p*$ on the universe W

defined as

$$\mu_{p*}(w) = \sup_{w' \models p} S(w, w')$$

This fuzzy set can be understood as "approximate p" since $\mu_{p*}(w) = 1$ for each w where p is true and $\mu_{p*}(w)$ is computed as the maximum similarity degree between w and the prototypes of p (the worlds w' where p is true).

3. For each pair of classical propositions, Ruspini defined two conditional measures:
   - Conditional consistency measure as $C_S(p \mid q) = \sup_{w' \models q} \sup_{w \models p} S(w, w')$
   - Conditional implication measure as $I_S(p \mid q) = \inf_{w' \models q} \sup_{w \models p} S(w, w')$

   The first measure is symmetric and it is the maximum membership degree of $p*$ restricted to the worlds where q is true (or symmetrically, the maximum membership degree of $q*$ restricted to the worlds where p is true) while the second is not symmetric and it is the minimum value of $p*$ restricted to the worlds where q is true (a kind of degree of inclusion of q in $p*$).

A similarity logic based on Ruspini's ideas was developped in [**?**] where it is also related to possibilistic and fuzzy truth value logics.

In this paper, a new fuzzy interpolation method based on the preservation of the generalization to fuzzy sets of the conditional measures given before, is defined.

2 Similarity-based method

2.1 Introductory notions

Given a fuzzy set A on the universe X and a similarity relation S on X, we can define the extension of A by S (see for example Jacas or) as the fuzzy set $S \circ A$ defined by $(S \circ A)(x) = \sup_{u \in X}(\min(S(x, u), A(u))$. In this paper we will use as similarity relations on a universe X of real numbers, the family S_r for all positive real number r (introduced by Godo and Sandri in [10]) and defined by $S_r(x_1, x_2) = \max(1 - \frac{|x_1 - x_2|}{r}, 0)$.

A triangular fuzzy set A is characterised by an ordered triple (a, b, c) with $a \leq b \leq c$ such that $[a, c]$ and $\{b\}$ are respectively the support and the core of A. Analogously, a trapezoidal fuzzy set A is characterised by an ordered quadruple (a, b, c, d) with $a \leq b \leq c \leq d$ such that $[a, d]$ and $[b, c]$ are respectively the support and the core of A.

Proposition 2.1 *(see [10]) Given a triangular fuzzy set A defined by $(a - q, a, a + k)$, the extension of A by the similarity S_r is defined by $(S_r \circ A)(x) = (a - q - r, a, a + k + r)$.*

On the other hand, conditional consistency and implication measure with respect to a similarity relation have been introduced by Ruspini in [12] for classical sets and fuzzy similarity relations as we explained in the introduction. For two classical sets A and B on a universe X, these conditional measures are defined as:

$$C_S(A \mid B) = \sup_{w \in A}(\sup_{w' \in B} S(w, w'))$$

$$I_S(A \mid B) = \inf_{w \in A}(\sup_{w' \in B} S(w, w'))$$

In the literature, there are different generalizations of this conditional measures to fuzzy sets but perhaps the most natural are the ones given below. Let A and B be fuzzy sets on the universe X and let S be a similarity relation on the same universe. Given a fuzzy implication $\rightarrow$, conditional measures can be defined by:

$$C_S(A \mid B) = \sup_{u \in X}(\min(A(u), (S \circ B)(u))) \quad (1)$$

$$I_S(A \mid B) = \inf_{u \in X}((A(u) \rightarrow (S \circ B)(u))) \quad (2)$$

It is not dificult to prove that C_S remains symmetric (in fact, it is equal to $\sup_{u,v \in X}(\min(A(u), B(v), S(u, v)))$ while it is not the case of I_S. In this paper, we will consider the case where $\rightarrow$ is the residuation of the minimum t-norm. In such a case, a simple computation shows that the values of conditional measures are:

$$C_S(A \mid B) = \sup_{u \in Supp'(A)} (S \circ B)(x)$$
$$I_S(A \mid B) = \inf_{u \in Supp'(A)} (S \circ B)(x)$$

where $Supp'(A) = \{x \mid A(x) > (S \circ B)(x)$ (see figure 1).

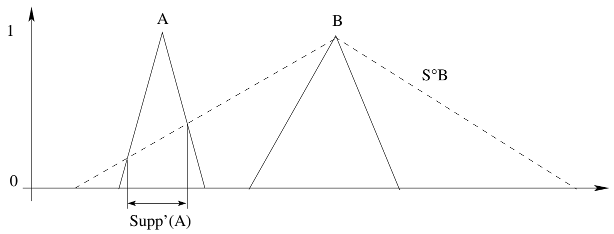

Figure 1: $Supp'(A)$

Notice that the value of the implication in formula (2) is 1 for points not belonging to $Supp'(A)$ and $(S \circ B)(x)$ for points belonging to $Supp'(A)$.

2.2 Similarity-based method for normalized triangular fuzzy sets

2.2.1 Description of the procedure

Let's go to describe the interpolation method proposed. First, we study the case of triangular fuzzy sets. Let $A_1 = (-h, 0, l)$, $A_2 = (a-q, a, a+k)$ and $A = (b-u, b, b+v)$, be triangular and normalized fuzzy sets corresponding to the premises of the rules and the input. Suppose that $a < b$ (A is in between A_1 and A_2). Suppose you rescale the univers V where the conclusion are defined in such a way that $Core(B_1) = \{0\}$ and $Core(B_2) = \{a\}$. After rescaling, suppose that the conclusions of the rules are the triangular and normalized fuzzy sets $B_1 = (h', 0, l')$ and $B_2 = (a-q', a, a+k')$. In such conditions, the method proposed is described in the following procedure (see figure 2):

1. Take on the input space the similarity relations S_t defined by Godo-Sandri for all t such that $C_{S_t}(A_2 \mid A_1) > 0$ and $I_{S_t}(A_2 \mid A_1) > 0$ and compute $C_{S_t}(A \mid A_1) = c_t$ and $I_{S_t}(A \mid A_1) = i_t$.

2. Compute for each t the values $m(t)$ and $n(t)$ such that $C_{S_{m(t)}}(B_2 \mid B_1) = C_{S_t}(A_2 \mid A_1)$ and $I_{S_{n(t)}}(B_2 \mid B_1) = I_{S_t}(A_2 \mid A_1)$.

3. Find points y_t and z_t on V such that $(S_{m(t)} \circ B_1)(y_t) = c_t$ and $(S_{m(t)} \circ B_1)(z_t) = i_t$.

4. Compute the equations of the lines defined by points (y_t, c_t) and (z_t, i_t) for all t. An easy computation shows that these lines are straight lines containing the point $(b, 1)$. If the triangle defined by these lines defines a fuzzy set, we take it as the output B' corresponding to the input A when conditioning by A_1. Of course, $Core(B') = \{b\}$ as in all known methods.

5. Repeat the process changing the roles of A_1 and A_2 and also of B_1 and B_2 and find B" which is taken, when defining a fuzzy set, as the output corresponding to the input A when conditioning by A_2.

Taking into account definitions of conditional measures and B', observe that the 3rd step of the above procedure assures that $C_{S_{m(t)}}(B' \mid B_1) = C_{S_t}(A \mid A_1)$ and $I_{S_{n(t)}}(B' \mid B_1) = C_{S_t}(A \mid A_1)$.

The procedure described before is sound due to the following results:

Proposition 2.2 *Using the notation of the procedure described before we have:*

1) Values t that satisfy item 1. of the procedure are the values $t > a - q - h$ for consistency measure and $t > a + k - l$ for implication measure.

2) If $r > 0$ and $t = (a + k - l) + r$, then $n(t) = (a+k'-l'+r)$ and if $r > 0$ and $t = (a-q+-l)+r'$, then $m(t) = (a-q'-l')+r'$.

3) Each family of points (y_t, c_t) and (z_t, i_t) for all t belongs to a straight line containing the point $(b, 1)$.

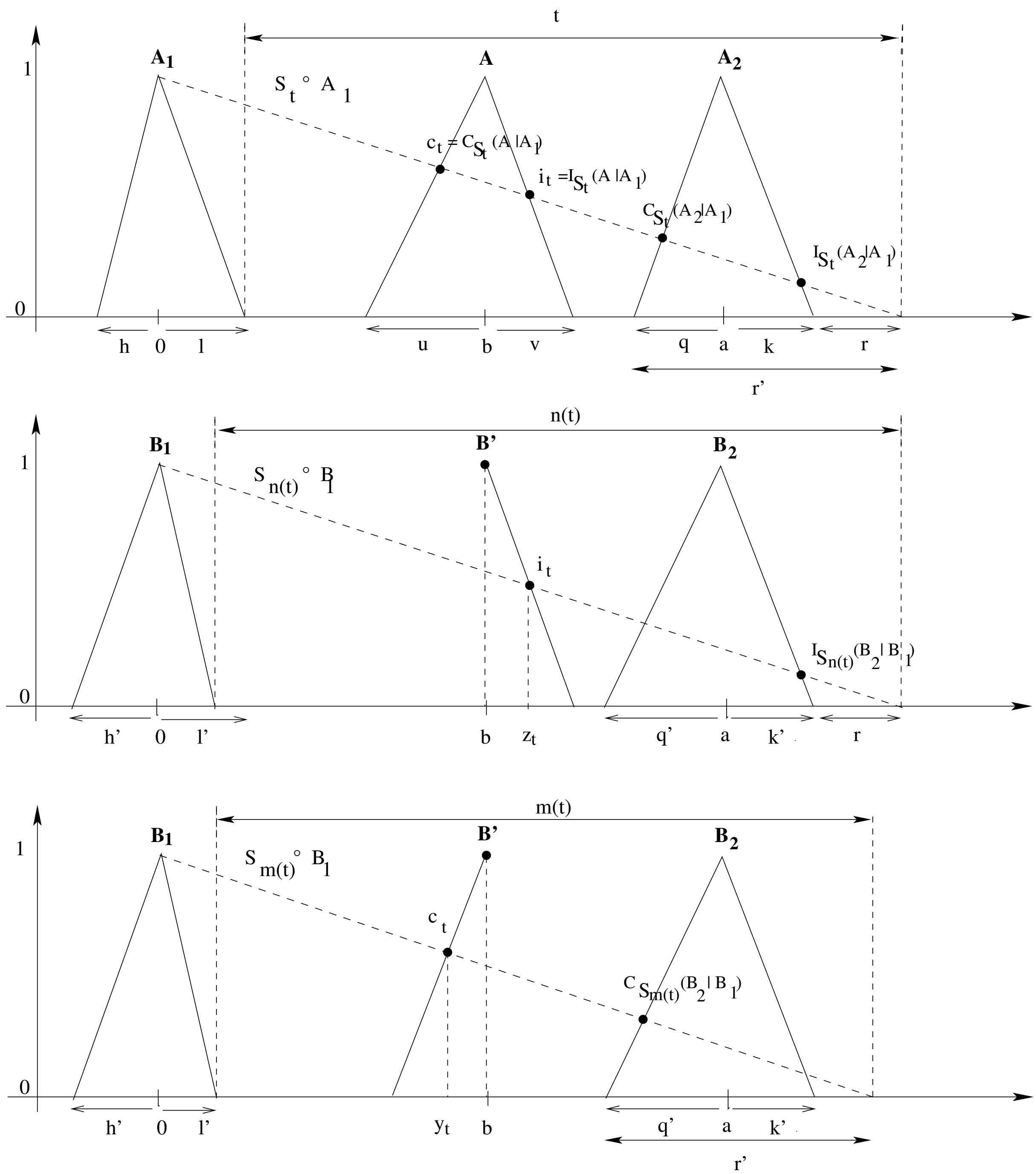

Figure 2: Similarity-based interpolation

4) B' is the triangle defined by the triple $(b-(u+q'-q), b, b+(v+k'-k))$

Proof. The results of point 1 is an easy consequence of the definition of the conditional measures and the result of proposition 2.1.
To prove 2 and 3 let us compute it for implication measure. The computations and results for consistency measures are obviously analoguous. To prove 2, notice that if $t = a+k-l+r$, then $(S_t \circ A_1)(x) = (-h-(a+k-l+r)), 0, l+(a+k-l+r)) = (-h-(a+k-l+r), 0, a+k+r)$ and that $I_{S_t}(A_2 \mid A_1)$ is the greater value of $S_t \circ A_1$ on the set $Supp'(A_2)$, i.e., the value of the intersection point of the right hand lines of $S_t \circ A_1$ and A_2 as we can see in figure 2. We can remark that:

Lemma 2.3 *Let l_1 and l_2 be the lines determined by:*

$$\begin{aligned} l_1 &: (0,1) \quad \textit{and} \quad (a+k+r, 0) \\ l_2 &: (a,1) \quad \textit{and} \quad (a+k, 0) \end{aligned}$$

for a, k and r real numbers. Then the intersection point is (α, β) where $\beta = \frac{r}{a+r}$.

From this lemma, the intersection of these two lines is $\frac{r}{a+r}$ and this is the value of $I_{S_t}(A_2 \mid A_1)$. Then 2 follows from the observation that this point depends only on the point a and the value of r and this is also valid in the output space with the obvious change of values (k' for k and l' for l). It is clear that for preserving I_S we need to preserve r while for preserving $C_S(A_2|A_1) = I_S(B_2|B_1)$ we need to preserve $r' = q+k+r$. Only when $q+k = q'+k'$ we have $n(t) = m(t)$. Moreover we can compute the value i_t taking the value of A at the greatest intersection point with $S_t \circ A_1$. This value is $i_t = \frac{a+k+r-b-v}{a+k+r-v}$. To compute points z_t such that $S_{n(t)} \circ B_1 = i_t$ take the equation of the right hand line of $S_{n(t)} \circ B_1$ which is $y = -\frac{x}{a+k'+r"} + 1$ and by an easy computation we obtain $z_t = b[\frac{a+k'+r}{a+k+r-v}]$.

Finally to prove that points (z_t, i_t) for all t belongs to a straight line containing the point $(b,1)$ take points for different values of t (for example $t = r$ and $t = 0$) and the point $(b,1)$ and an easy computation shows that these three points are in the same straight line for any value of r. One way is to compute that the value of the determinant of dimension 2 of the differences of the two first points with $(b,1)$ is 0 (assuring that the vectors determined by points $(b,1)$ and (z_r, i_r) and by $(b,1)$ and (z_0, i_0) are lineraly dependent. ■

2.2.2 Aggregation of results

From results of proposition 2, we can compute the straight lines defining the triangles B' and B''. The computation shows that the triangles, solutions of this (similarity-based) interpolation method, are B' defined by $(b-(u+q'-q), b, b+(v+k'-k))$ and B'' defined by $(b-(u+h'-h), b, b+(v+l'-l))$. Surprisingly these triangles coincide with the ones obtained by Bouchon et al. in [7] by a very different based method. Moreover B' and B'' are fuzzy sets, if and only if, $u > \max(h'-h, q'-q)$ and $v > \max(k-k', l-l')$. In bad cases (cases such that the solution is not a fuzzy set) it is obvious that we could find the most similar input (to A) which corresponding outputs by (similarity-based) interpolation are a fuzzy set. Obviously, if we are in a bad case, we always could take the (more imprecise) input A' defined by $(b-\max(u, h'-h, q'-q), b, b+\max(v, k'-k, l'-l))$ in order to obtain a convex outputs B'_1 and B''_1. This coincide also with what is done in Bouchon et al. in [7] for obtaining the so-called convex fuzzy sets in cases that B' and B'' are not so. Moreover for this "bad" cases the method may be improved by using (like in [5]) some similarity measure to compute the similarity between A and A' and to obtain the outputs by modifying B'_1 and B''_1 accordingly. Finally to compute the result (supposing you have obtained as results two fuzzy sets B' and B'') we need to aggregate them and we can use different aggregation functions, for example the one used by Bouchon et al. in [7] where $B = \mu B' + (1-\mu)B''$ with $\mu = \frac{a-b}{a}$.

2.2.3 Remarks

1) Similarity based method return $B' = B'' = B_1$ if $A = A_1$ and $B' = B'' = B_2$ if $A = A_2$ and it returns only (B' is not computable) a proper subset B'' of B_1 if A is a proper subset of A_1. Similarly it returns only a proper subset B' of B_2 if A is a proper subset of A_2.

2) A possible drawback of the method is the fact that the output triangles B' and B'' do not vary the shape when the input A has a fixed triangular shape and it is moving from having the same core than A_1 to have the same core than A_2.

3) Notice that points (y_t, c_t) and (z_t, i_t) do not cover all the points of the straight lines defining the triangle B' and the same is true for the lines defining B''. For example points (z_t, i_t) cover only the segment of the right hand straight line of B' that goes from the straight line $y = [-\frac{x}{a+'}] + 1$ till the point $(b, 1)$.

4) Using only the conditional implication measure we would have obtained a triangle which right hand line had been computed conditioning by A_1 and the left hand line conditioning by A_2. The resulting triangle would be $B = (b-(u+l'-l), b, b+(v+k'-k))$ which defines a fuzzy set if and only if $u > l - l'$ and $v > k - k'$. In the same way using only the conditional consistency measure we would have obtained the triangle $B = (b-(u+h'-h), b, b+(v+q'-q))$. The conditions for being a fuzzy set could be given analogously.

3 Comparison with related methods of fuzzy interpolation

We propose to compare our method with the Dubois and Prade's method [3] and Kockzy et al.'s [2] method. The comparison can be achieved on the basis of two criteria: convexity and specificity of solution.

4 Conclusion

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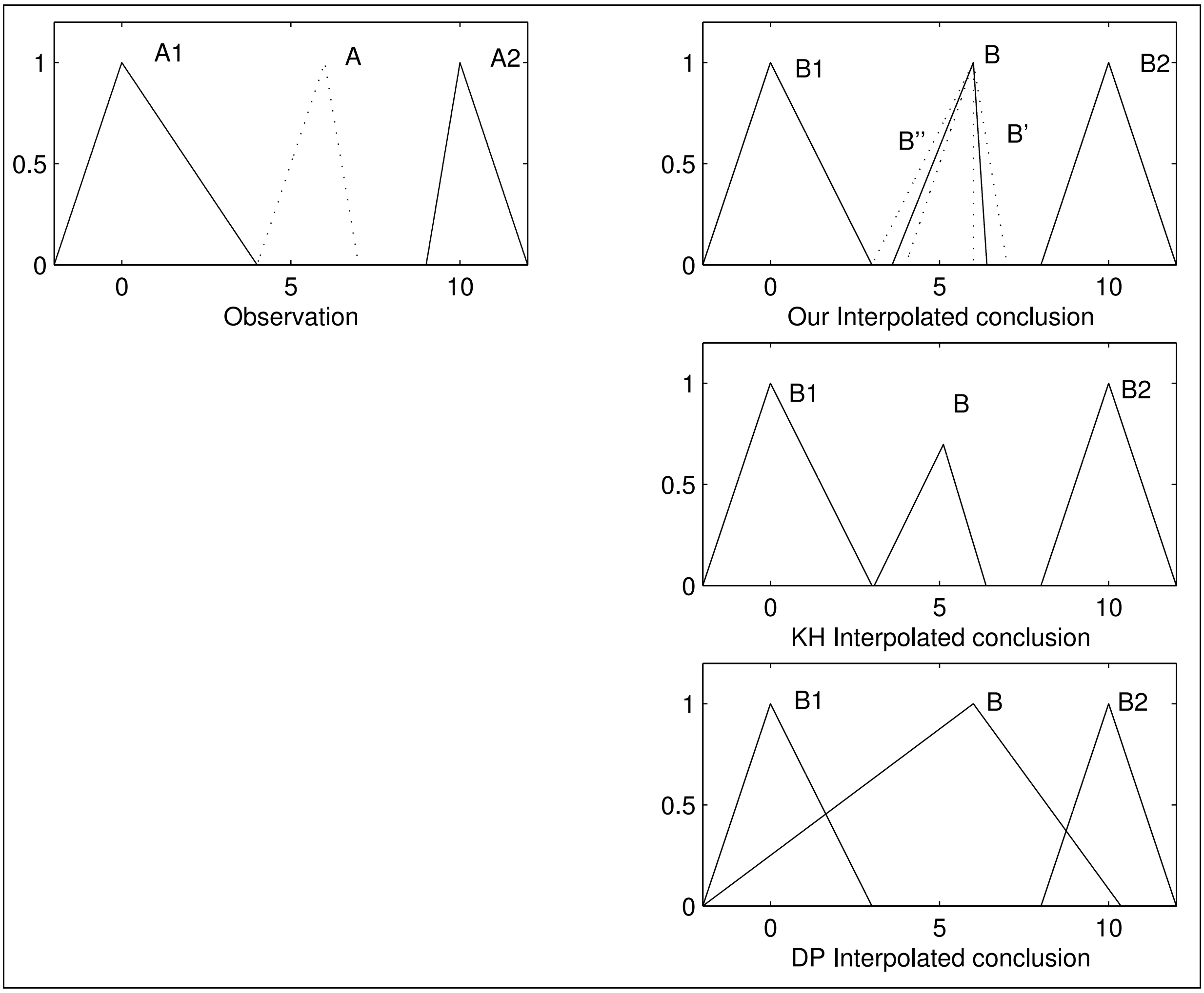

Figure 3: Comparison with Dubois and Prade's and Kóckzy et al.'s methods of interpolation

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