# On the paradoxical success of Mamdani's minimum-based inference

M. Detyniecki and B. Moubêche and B. Bouchon-Meunier

**Abstract** Mamdani's inference has an incredible success, especially in areas such as decision making and control. Yet, it is well known that it uses a min-based implication that does not verify classical boolean logic requirements.
This contribution aims at, in the one hand, exploring Mamdani's choice from a practical point of view, and in the other hand, explaining the success of Mamdani's inference from a logical perspective, by introducing a simple variant of the Generalized Modus Ponens (GMP) that uses standard fuzzy implications.
In addition, this new formulation opens the way for new methods of inference that have the same benefits as Mamdani's.

## 1 The success

Fuzzy logic control has been very successful since its introduction in the second half of the XX$^{\text{th}}$ century. Many technologies are based on fuzzy logic control, ranging from large projects such as the Sendai subway system [14], complex ones as unmanned helicopter control [13] to everyday life tools, such as washing machines [1]. Fuzzy control is based on inference rules that, based on observed information, produce regulatory decisions. The reasoning uses the Modus Ponens principle to transform these observations into actions. Several of such control architectures exist,

Marcin Detyniecki
Laboratoire d'Informatique de Paris VI, CNRS-UPMC, 4, place Jussieu, 75 005 Paris, France, e-mail: marcin.detyniecki@lip6.fr

Benjamin Moubêche
ENS Cachan, Département de Mathématiques, 61, avenue du Président Wilson, 94 230 Cachan, France, e-mail: benjamin.moubeche@ens-cachan.fr

Bernadette Bouchon-Meunier
Laboratoire d'Informatique de Paris VI, CNRS-UPMC, 4, place Jussieu, 75 005 Paris, France, e-mail: bernadette.bouchon-meunier@lip6.fr

most popular being the one proposed by Mamdani in 1975 [10], [11]. Strangely, what we could call "Mamdani's Modus Ponens" uses a minimum-based pseudo-implication, which is well known not to meet boolean logic requirements.

This contribution aims at explaining the success of Mamdani's inference though it is not based on a standard fuzzy implication, by introducing a simple variant of the Generalized Modus Ponens (GMP) with standard fuzzy implications. We begin by exploring, in section 2, the particularities of Mamdani's approach. Then in section 3 we propose a GMP integrating the observation-premise compatibility, that will not only explain Mamdani's reasons but also open the framework to new formulations, presented in section 3.3.

The use of the minimum in the place of a standard implication (i.e. fuzzy implication that behaves like the classical implication, when the membership values are boolean values) has been discussed and explained by several authors [20], [5], [9]. Among the answers, it is accepted that this particular choice corresponds to an interpretation of the underlying relation consistent with the cartesian product.
In this paper we do not dispute any of the known conclusions, we aim at providing an alternative explanation from a logical calculus perspective.

## 2 The Generalized Modus Ponens (GMP) and Mamdani's controller

### *2.1 Notations*

The classical Modus Ponens is used to draw conclusions from pre-established rules. Thus, from an implication $A \Rightarrow B$ ("*If* the light is red, *then* stop"); and given the premise $A$ ("the light is red"), we can deduce $B$ ("stop").

$$A \wedge (A \Rightarrow B) \rightarrow B \tag{1}$$

In general, the observations are more or less true, according to the certainty that can be attached to the observation. It is therefore important to be able to reason with a "modified" observation. Approximate reasoning introduced by Zadeh [21] relaxes the classical form, by dealing with observations that are not necessarily exactly equal to $A$. The Generalized Modus Ponens can leverage the underlying implication [5], [15] and uncertainties of a premise close to $A$, denoted $A'$, to infer a conclusion $B'$, not necessarily equal to $B$.
Thus, Zadeh proposes to extend the Modus Ponens to any type of observations.
More precisely, the membership function of $B'$ is calculated using the formula:

$$\mu_{B'}(y) = sup_{x \in X} T(\mu_{A'}(x), R_I(x,y)) \tag{2}$$

where $\mu_{B'}(y)$ is the membership value of the conclusion set $B'$ obtained by the Modus Ponens at set item $y$. $\mu'_{A'}$ is the membership function of the observation set $A'$, $T$ a t-norm and $R_I$ a fuzzy implication.

Knowing that the t-norms are used as conjunction operators, the previous equation can be seen as an extension to fuzzy sets of the following tautological formulation:

$$A' \wedge (A \Rightarrow B) \rightarrow B' \tag{3}$$

We notice that the *sup* corresponds to the disjunction of all possibilities [9], since we are reasoning on sets.
Moreover, we obtain the classical formulation (1) in the case where $A'$ is identical with $A$.

## 2.2 Mamdani's controller

Proposed in 1974 to control a steam engine [11], Mamdani's control is based on a formulation analogous to the Generalized Modus Ponens, and since then used especially in the field of control [4], [2], [3]. The first stage of such a controller, called fuzzification [4], consists in obtaining data from the observations and translating them into fuzzy sets. These observations will then be processed by a set of rules governing the controller. It is during this phase, known as Mamdani's inference, that the conclusion $B'$ is computed using the following formula:

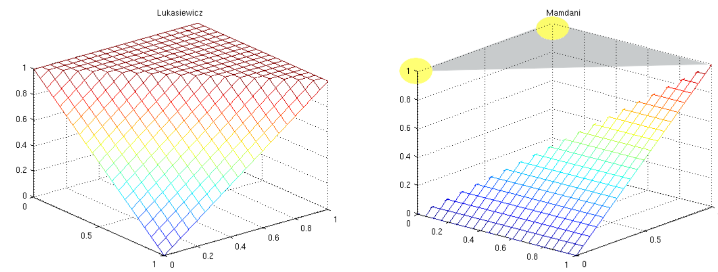
$$\mu_{B'}(y) = sup_{x \in X} min(\mu_{A'}(x), min(\mu_A(x), \mu_B(y))) \tag{4}$$

Then, the conclusions obtained by each rule are then aggregated disjunctively. Since we are in a control situation and action is necessary, the final set is then translated into a single real value. Mamdani chose the center of gravity of the set. This last step is called defuzzification.

## 2.3 The paradox

We note that formula (4) corresponds to the Generalized Modus Ponens (2) when choosing the *min* as t-norm and again the *min* as implication. Yet, it is well known that the *min*, even though called by several authors "Mamdani's implication", is *not* a standard fuzzy implication.
Several differences are notable, in particular, when the premise is totally false (i.e. boolean case). In fact, in that case "Mamdani's implication" will false the statement, while a standard fuzzy implication will return total truth, as shown in Figure 1.

If we look at the domain covered when using fuzzy logic (i.e. $[0,1] \times [0,1]$), "Mamdani implication" differs from a standard fuzzy implication in the area where




(a) Lukasiewicz implication [17] (b) "Mamdani implication"

**Fig. 1** Graphical comparison between a standard fuzzy fuzzy implication and Mamdani's approach.

the second variable (logical consequent) is greater than the first one: shaded area in Figure 1). In fact, instead of giving a high truth value for the implication, the result will be low (or zero). These differences will have a strong impact on the behavior of "Mamdani's GMP" compared to others, especially when the observation set $A'$ is *very* different (or even *disjoint*) from $A$.

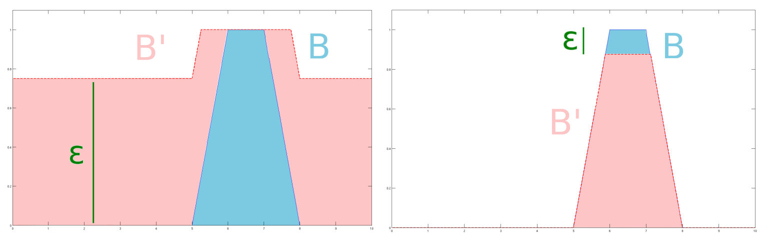
## *2.4 Differences in uncertainty management*

Depending on the chosen t-norm/implication combination (2), the uncertainty will not be managed in the same way [6], [4]. Moreover, not all combinations are possible. Several constraints may be imposed for the choice of operators. Typically, only the "identity property" is required, that is to impose to recover $B$ when $A = A'$. The main possible combinations [6] under this constraint are listed in Table 1.

**Table 1** Common viable operator combinations for the classical GMP.

| **t-norm** | **compatible implications** |
|---|---|
| Lukasiewicz<br>$T(u,v) = max(u+v-1,0)$ | Lukasiewicz<br>Kleene-Dienes<br>Wilmott<br>Bouwer-Gödel<br>Reicher-Gaines |
| Zadeh<br>$T(u,v) = min(u,v)$ | Bouwer-Gödel<br>Reicher-Gaines |

Most combinations [9] will place a flat and extended level of uncertainty $\varepsilon$ outside of the set (see Figure 2(a)), so that in the limit case where the two sets $A$ and $A'$ are disjoint, the universe of the conclusion will be all covered by a degree of membership equal to 1. In contrast, Mamdani's approach, as defined in 2.2, thresholds the values of the set (see Figure 2(b)). Here, for the disjoint case, we obtain a value of 0 for the whole universe.




(a) Lukasiewicz t-norm and implication (b) Mamdani's approach

**Fig. 2** Differences in uncertainty management between inference models.

More precisely, we can say that these "flat levels" of uncertainty come from the use of the implication. Indeed, as we have already seen, when the degree of truth of the premise $A$ is larger than the one of the conclusion $B$, the *min* does not act like other implications. When the observation $A'$ is outside the support of $A$, Mamdani's implication will zero any output out, since it rejects the idea that the implication based on a false premise can be true. But if we were using a standard fuzzy implication, we would have a high value and using the *sup* we would obtain the overall flat level of uncertainty.

### *2.5 Mamdani's choice*

The fundamental differences, described above, can have consequences that go beyond the interpretation given to the known behavior. In the case of control (and also of decision making), the disjoint case is a particular challenge when non-Mamdani's inference is used. In particular challenging problems are the defuzzification and the combination of results of several rules - both fundamental steps in control. The disjoint case may seem a limit case from a theoretical perspective, but in practice it is rather often the case. In fact, noise or any non relevant information (for the rule) will produce a disjoint observation.

If the sets $A$ and $A'$ have disjoint supports, the result with a standard fuzzy implication will *only* depend on the *defined universe* of the conclusion. More precisely, as follows from above, the conclusion set will be the universe with full membership (equal to 1). As a consequence, a value obtained by any defuzzification will be

meaningless.
If we ignore the problem of infinity inherent to certain universes of definition, as for instance distances, the problem still remains. For instance, let us consider in a control problem a rule that concludes with an angle to be chosen for an action. Then, if the measure is out of scope of the hypothesis, the whole universe (from 0° to 360°) will have a degree of membership equal to 1 (with an approach different from Mamdani's). This will imply, after a gravity based defuzzification, a degree of 180°, whatever the rule.

If Mamdani's approach is used, we would obtain as conclusion an empty set, which by any defuzzification technique would lead, in all cases, to an impasse (e.g. we do not know how to defuzzify), but not to a random value (as it may happen in the other approaches). It can be argued that the problem, in this extreme case, is ill defined and defuzzification should not be applied, but the struggle will still remain in a certain degree, for the approaches different to Mamdani's, depending on the extent to which the sets $A$ and $A'$ are different. In fact, in less extreme cases, we either have an infinity domain or a bounded universe, which will influence the defuzzification, unless one chooses to defuzzify by using the mode [4] (i.e. by using the point where membership is maximal). In fact, this approach is independent of the universe, but unfortunately reduces the entire fuzzy treatment (during the inference) to a reasoning on crisp values.

The fact that rules are applied for observations that are out of scope has been already noticed by Moser et al. [12], who propose to check the support of the rules to condition their activation. This solves the totally-disjoint case, but not the "almost" disjoint one.

In classical fuzzy control systems this problem is generally avoided by assuming that in a large set of rules, at least one pertinent rule will be fired producing a less uncertain conclusion, which - by *conjunctive* aggregation [8], [9] - will mask the above mentioned phenomena. Conjunctive aggregation has been identified already in the early days of fuzzy control [18] as being the proper way when standard implications are considered.

The debate of which aggregation framework (conjunctive or disjunctive), and more generally whether Mamdani's control or an alternative, is better, is out of scope of this paper. The facts are that both have proven to be very successful. Here, we aim at exploring the possible reasons of Mamdani's choice.

Mamdani's inference follows the disjunctive utilization of rules in classical propositional logic, which is used, for instance, in expert systems or in logical programming. But, if a standard implications is combined to a *disjunctive* aggregation, several difficulties would appear when aggregating inference rules. When at least one observation is out of the scope of any of the available rules - even if in the same domain there is a rule with a premise that applies - the conclusion obtained by a non-Mamdani inference will produce a conclusion that dominates, disjunctively, all other conclusions (as shown in Figure 3). One conclusion will cover the space and by no means any other rule will be able to unlock the situation.

To fully appreciate the advantage of Mamdani's approach, let us consider a almost disjoint observation. Then the conclusion fuzzy set will be, in Mamdani's case, an almost empty set with strictly positive membership inside the support of the original conclusion, while, in the other cases, we will have a non-null membership all over the defined universe, which will remain even when conjunctive aggregation is used, introducing a similar universe-definition-bias (in a smaller degree) as the one we mentioned in the case of a single rule.

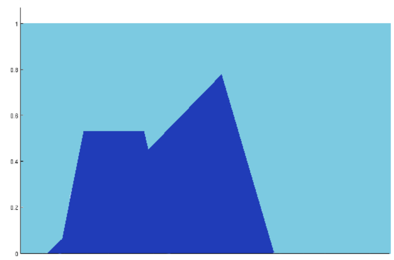


**Fig. 3** In a disjunctive aggregation framework a conclusion $B'_2$ (in light) obtained with a standard implication for out of scope observation, will dominate any other conclusions $B'_1$ (in dark) obtained so far.

Finally, we can say that Mamdani's approach has the indisputable advantage of being able to handle *alternative* (disjoint) conclusions, avoiding to cancel them out with an conjunctive aggregation. Tthis framework handles naturally the problem of out of scope observations (or noise), by producing empty set conclusions. The disjunctive aggregation of these sets, will *de facto* ignore the conclusions of the rules for which the Modus Ponens should not have been, anyway, applied.

Paradoxically, Mamdani's inference has several logical advantages that come from the use of the *min* instead of a standard fuzzy implication.

## 3 GMP with observation-premise compatibility (GMP-OPC)

In the classical case, when the observation does not match the premise, the Modus Ponens does not apply. By analogy, when the sets $A$ and $A'$ are disjoint, the GMP should not apply. In the same spirit, we propose a new formulation of the Generalized Modus Ponens with observation-premise compatibility, based on the following framework:

$$(A \wedge A') \wedge (A \Rightarrow B) \rightarrow B' \tag{5}$$

We note that this formulation is consistent with the classical Modus Ponens, since we find (1) when $A = A'$.

We propose to extend the previous logical framework to a set formulation, using an element-based interpretation of the sets, which gives a mathematical formula based on the *sup* and which is similar to the GMP formula (2).

$$\mu_{B'}(y) = sup_{x \in X} T(min(\mu_A(x), \mu_{A'}(x)), R_I(x,y)) \tag{6}$$

Technically, we measure the compatibility by the conjunction of observation $A'$ and premise $A$. It can be shown that, if we expect that $A' = A$ implies $B' = B$ then the only viable t-norm is the minimum.

## *3.1 Comparaison of GMP-OPC and Mamdani's approach*

Using a semi-automatic demonstration process we have shown that the GMP-OPC as defined in equation (6) is equivalent to the Mamdani's inference (equation (4)), for the combinations (of t-norms $T$ and implications $R_I$) listed in Table 2.
In other words, Mamdani's approach, with an implication that is not a standard one, corresponds to General Modus Ponens with an observation-premise compatibility, even if it is not obvious to see it, in the formulation.

**Table 2** Combinations of operators for which the GMP with observation-premise compatibility is equivalent to Mamdani's inference.

| **t-norm** | **implication** |
|---|---|
| Lukasiewicz | Lukasiewicz<br>Bouwer-Gödel<br>Reicher-Gaines |
| Zadeh | Bouwer-Gödel<br>Reicher-Gaines |

Because of the large number of cases appearing (due to the *max* and *min* in the formulations), we opted for a semi-automatic approach. A set of premises $A$ was defined, followed by a complete exploration of all possible configurations of $A'$. Finally, results from GMP-OPC (for each combination in table 2) and Mamdani's approach were compared.

## 3.2 Properties

For any t-norm $T$ and any implication $R_I$, formulation (6) exhibits a certain number of properties:

#### 3.2.1 Property for $A \cap A' = \emptyset$

If $A$ and $A'$ are disjoints, then $B' = \emptyset$
Proof :
Since $A$ and $A'$ are disjoints, $\forall x \;\; min(\mu_A(x), \mu_{A'}(x)) = 0$
Then, by introducing this in equation (6), we obtain

$$\forall y \;\; \mu_{B'}(y) = sup_{x \in X} T(0, R_I(x,y)) = 0 \tag{7}$$

Which is the definition of the empty set.

#### 3.2.2 Property for $A' \subset A$

If $A'$ is included in $A$, then the GMP and the GMP-OPC are equivalent.
Proof :
By definition of the inclusion $\mu'_A(x) \leq \mu_A(x)$ Thus,

$$min(\mu_A(x), \mu_{A'}(x)) = \mu_{A'}(x) \tag{8}$$

If we replace equation (8) in equation (6) : GMP-OPC, we obtain

$$\mu_{B'}(y) = sup_{x \in X} T(\mu_{A'}(x), R_I(x,y)) \tag{9}$$

Which is nothing else than equation (2) of the classical GMP.

#### 3.2.3 Property for $A' \supset A$

If $A'$ contains $A$ and $T$ and $R_I$ are compatible for the classical GMP (as defined in section 2.4), then $B' = B$
Proof :
The fact that $A'$ contains $A$ implies by definition that $min(\mu_A(x), \mu_{A'}(x)) = \mu_A(x)$
Replacing in equation (6), we obtain:

$$\mu_{B'}(y) = sup_{x \in X} T(\mu_A(x), R_I(x,y)) \tag{10}$$

which is equal to equation (2) when $A' = A$, implying by definition of the compatibility that $B' = B$, since $T$ and $R_I$ were chosen to have a compatible GMP.

### 3.2.4 Property for $A' = A$

If $A' = A$ and $T$ and $R_I$ are compatible for the classical GMP (as defined in section 2.4), then $B' = B$
Proof :
Since $A' = A$, we also have $A' \subset A$ and property 3.2.2 applies (i.e. GMP-OPC is equivalent to the classical GMP). Since $T$ and $R_I$ are compatible for the classical GMP, we have by definition $B' = B$.

### 3.2.5 Viability of combinations $T$ and $R_I$ for the GMP-OPC

The combination of operators $T$ and $R_I$ viable for the GMP with observation-premise compatibility, are exactly the ones of the GMP (table 1).
Proof :
If $A' = A$, then we also have $A' \subset A$ and thus property 3.2.4 applies, which implies that in that case ($A' = A$), the classical GMP and the GMP-OPC are equivalent. Since the viability is defined when $A' = A$, the equivalence applies.

## 3.3 New GMP-OPC inferences

It is generally accepted [6] that the only necessary and essential property to declare a formulation of the GMP as viable is the satisfaction of the "identity property" (as recalled in section 2.4). Taking into account property 3.2.5, by comparing Table 1 (on the operators combination viability) with Table 2 (on the equivalence between GMP-OPC and Mamdani's inference), we notice that several new formulations are not equivalent to Mamdani's approach, but are viable. These new formulations propose variants of the most popular inference method in the world of applications, all keeping its intrinsic advantages.

For example, if we choose as $T$, the Lukasiewicz t-norm [17] and as $R_I$, the Kleene-Dienes implication [19], we obtain the following formula:

$$\mu_{B'}(y) = \sup_{x \in X} \max(\min(\mu_A(x), \mu_{A'}(x)) + \max(1 - \mu_A(x), \mu_B(y)) - 1, 0) \qquad (11)$$

This new form of GMP with observation-premise compatibility not only satisfies, by definition, the properties of section 3.2, but also addresses the drawback of the classical GMP stated in section 2.5. The difference between Mamdani's approach and this new formulation (as shown on Figure 4) is that the new version may conclude with a set $B'$ more precise than $B$.

If we now choose Wilmott's implication [16], with the Lukasiewicz t-norm [17], we obtain the following formula:

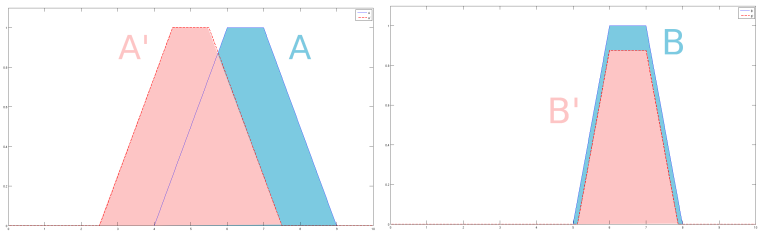



**Fig. 4** Result of GMP with applicability of the rule for the Lukasiewicz t-norm and Kleene-Dienes implication.

$$\mu_{B'}(y) = \sup_{x \in X} \max(\min(\mu_A(x), \mu_{A'}(x)) + \max(1 - \mu_A(x), min(\mu_A(x), \mu_B(y))) - 1, 0) \tag{12}$$

As before this formulation has the properties of section 3.2, adresses drawbacks of section 3.2 and as shown on Figure 5 provides a conclusion $B'$ more precise than $B$, but this time with some extra uncertainty compared to the previous example.

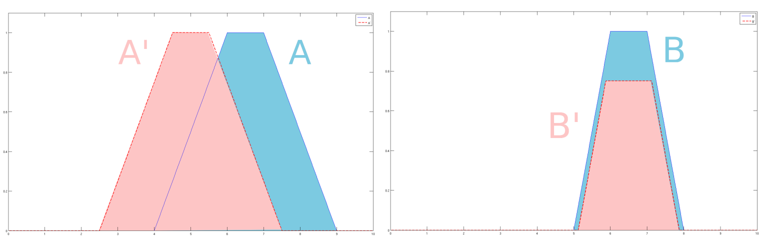



**Fig. 5** Result of GMP with applicability of the rule for the Lukasiewicz t-norm and Wilmott implication.

## Conclusion

Mamdani's inference has had and still has an incredible success, especially in areas such as decision making and control. Yet it is well known that it uses a min-based implication that does not satisfy even classical logic requirements. Our study shows that the fundamental difference between the standard Generalized Modus Ponens and Mamdani's approach comes from set elements for which the observation and the premise of the rule are disjoint.

Mamdani's approach has proven its success with a vast number of application, not only in control but also in other domains such as decision making. The paradox discussed in this paper does not imply that Mamdani's approach is "wrong" (neither that it is better than an other one).

In this work, we propose a new Generalized Modus Ponens that includes in its formulation the applicability of the rule, by combining conjunctively the observation and the premise. And, we show that Mamdani's inference, that uses a minimum instead of a standard implication, is nothing but a Generalized Modus Ponens that takes into account the compatibility between the observation and premise.

It is remarkable that this is true for a large number of compatible t-norms and standard fuzzy implications. In addition, this new formulation opens the way for new methods of inference that have the same benefits as Mamdani's. In future works, these variations will be studied in more detail.

## References


1. R. Narayanan. *Fuzzy Logic Control of Washing Machines*, Electronic For You Magazine, India, December 2003.
2. P.P. Bonissone. *Fuzzy logic controllers: an industrial reality* In J.M. Zurada, Robert J. Marks II, and Charles J. Robinson, editors, Computational Intelligence: Imitating Life, pages 316-327. IEEE Press, Piscataway, NJ, 1994.
3. K. Hirota, editor. *Industrial Applications of Fuzzy Technology*. Springer, Tokyo, 1993.
4. R. Yager and D. Filev. *Essentials of Fuzzy Modeling and Control*, NewYork, USA: John Wiley and Sons, 1994.
5. C. Alsina, S. Guadarrama, E. Renedo, E. Trillas, *Studying Fuzzy Modus Ponens*, In Proceedings of the Annual meeting of the North American Fuzzy Information Processing Society (NAFIPS), pages 426-429, Montreal, Canada, 2006.
6. B. Bouchon-Meunier. *La logique floue et ses applications*. Addison-Wesley, 1995.
7. E. Trillas and L. Valverde, *On modus ponens in fuzzy logic*, Proceedings of the 15th International Symposium on Multivalued Logics (ISMVL), pages 294-301, Kingston,Canada, 1985.
8. D. Dubois, H. Prade. *Combination of fuzzy information in the framework of possibility theory*. In M.A. Abidi and R.C. Gonzalez, editors, Data Fusion in Robotics and Machine Intelligence, pages 481-505 . Academic Press, Inc ., Boston, 1992.
9. E. Trillas, *On logic and fuzzy logic* in International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 1, no. 2, pages 107-137, 1993.
10. E.H. Mamdani, S. Assilian. *IAn experiment in linguistic synthesis with a fuzzy logic controller* in International Journal of Man-Machine Studies, vol. 7, pages 1-13, 1975.
11. E.H. Mamdani. *Application of fuzzy algorithms for control of simple dynamic plant* in Proceedings of the Institution of Electrical Engineers (IEE), vol.121, no. 12, pages 1585-1588, 1974.
12. B. Moser, M. Navara. Fuzzy controllers with conditionally firing rules, *IEEE Transactions of fuzzy systems*, vol. 10, no. 3, pages 340-348, Jun 2002
13. M. Sugeno, M. F. Griffin, and A. Bastian, *Fuzzy hierarchical control of an unmanned helicopter*. In Proceedings of the 17th IFSA World Congress., pages 179182, Korea, 1993.
14. T. Terano, K. Asai, M. Sugeno. *Applied fuzzy systems*. Academic Press, Inc., Boston, 1994.
15. E. Trillas, M. Mas, M. Monserrat and J. Torrens, *On the representation of fuzzy rules*. in International Journal Approximate Reasoning, vol. 48, pages 583597, 2008.
16. R. Wilmott, *Two fuzzier implication in the theory of fuzzy power sets* in Fuzzy Sets and Systems, vol. 4, pages 3136, 1980.
17. J. Lukasiewicz, *Interpretacja liczbowa teorii zdan*, in Ruch Filozoczny, vol. 7, pages 93-93, 1923.

18. M. Mizumoto, *Fuzzy control under various fuzzy reasoning methods*, in Proceedings of the first IFSA World Congress, Mallorca, Spain, 1985.
19. Z. Dienes. *On an implication function in many-valued systems of logic*. in Journal Symbolic Logic, vol. 14, pages 9597, 1949.
20. B. Bouchon-Meunier, D. Dubois, L. Godo and H. Prade, *Fuzzy sets and possibility theory in approximate and plausible reasoning* in Fuzzy sets in approximate reasoning and information systems (vol. 5), part of the Handbook of Fuzzy Sets and Possibility Theory (J. Bezdek, D. Dubois, H. Prade, eds.), Kluwer Academic Publisher, pages 15-190, 1999.
21. L.A. Zadeh. *The Concept of a Linguistic Variable and its Application to Approximate Reasoning*, in Information Sciences 9, pages 43-80, 1976.