Double-Linear Fuzzy Interpolation Method

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Abstract—In this paper, we present an original fuzzy interpolation method. In contrast to existing proposals, our method is able to always construct an interpolated fuzzy interval without a need of a special step dedicated to a “standardization” of undesired obtained solutions which has less to do with interpolation. Furthermore, we show that the proposed method integrate the deviation to the expected linearly interpolated solution and constrain the constructed solution to extreme cases. We illustrate the behaviour of our method by a comparison to other well known fuzzy interpolation methods and we discuss these comparisons.

I. INTRODUCTION

Interpolation of fuzzy data was intensively studied since the first paper proposing a method [1] (or more accessible [2]). Many different points of views were adopted (the interested reader can find a quasi exhaustive list of references in [3] or [4]): α cuts approach (recent ones proposed in [5], [6] or [7]), analogy or similarity based approach ([8], [9]), [10], logical approach, etc. Some methods suppose that fuzzy data are represented only by trapezoidal fuzzy intervals, some others are more general and propose methods applicable to any membership function. Frameworks were proposed to compare different existing methods [3], [11], [12], or to unify them [4], [13]. The latter framework is an interesting theoretical approach enabling to write many methods in a unique way thanks to an analytic approach, but it is not easy to instantiate the different behaviours in concrete situations.

Usually, the methods are all agree on the position of the interpolated conclusion. But, they all have to deal with the problem to provide a suitable shape and with the possibility to obtain an undesired non convex fuzzy set. The reason of this problem (as it will be shown) is that the shape of the fuzzy numbers can not be linearly interpolated. The use of a threshold, or a standardization step dedicated ensure that the obtained membership functions are normalised and functions, helps usually to eliminate these situations. This is one of the major differences with the method we propose. Indeed, our aim is to propose a method without such wise trick. We propose to first linearly interpolate the position. Second to linearly interpolate the deviation between the observed form and the expected shape.

Another major characteristic of our method lies in the consideration of the deviation of the observation to the linearly interpolated fuzzy set.

Our paper is organised as follows. We give the notations in Section II before describing our method which is divided in two phases: the interpolation of the position phase is described in Section III and the interpolation of the shape phase is described in Section IV. In Section VI, we present some graphical comparisons with known methods for enhancing their different or common behaviours and we propose a discussion in Section VI. Finally, we conclude and give our future works.

II. INTERPOLATIVE REASONING AND NOTATIONS

A. Interpolative reasoning general principles

Let us consider two numerical variables $X$ and $Y$ defined on the universe $R$ of real numbers. Let $F$ denote the set of fuzzy sets $R$. We suppose that we are given fuzzy sets $A_i$ in $F$, $1 \leq i \leq n$, such that: $A_1 \preceq A_2 \cdots \preceq A_i \preceq A_{i+1} \cdots \preceq A_n$, for a given order $\preceq$ on $F$. We suppose also that we are given fuzzy sets $B_i$ in $F$, $1 \leq i \leq n$, which are also ordered according to $\preceq$.

The context of study concerns sparse fuzzy rule-based systems where fuzzy rules are of the type : $(R_i)$: “if $X$ is $A_i$ then $Y$ is $B_i$”. The sparsity of the system means that the premises of the rules do not cover the input space $F$ and there exist inputs $A_i$ such that $\forall/A_i \preceq A_s \preceq A_{i+1}$ and the support of $A_s$ does not intersect neither the kernel of $A_i$ nor the kernel of $A_{i+1}$.

The aim of a fuzzy interpolation method is to provide the conclusion corresponding to the observation $A_s$ by considering only the two rules $R_i$ and $R_{i+1}$ when $A_i \preceq A_s \preceq A_{i+1}$.

B. Notations and hypotheses

In this paper we focus on trapezoidal fuzzy sets. We choose to describe such a fuzzy set, as for instance $A_1$ shown on Fig. 1, with four parameters: - the length of the kernel, as computed by:

$$A_1^N = c - b \quad (1)$$

- the extend of the uncertainty on the left and on the right, defined by:

$$A_1^L = b - a \quad (2)$$

- the extend of the uncertainty on the left and on the right, defined respectively by:

$$A_1^R = d - c \quad (3)$$

and its position defined as the center of its kernel:

$$A_1^P = b + \frac{A_1^N}{2} = \frac{b+c}{2} \quad (4)$$

...
method will suffer from this non-linearity.

As for instances a length descriptor. Thus, any description of the shape but also to any linear shape measure which has different shape values. This counterexample exists.

A then build a fuzzy set observation is by, first computing the expected shape value (linearly) and then build a fuzzy set observation. Analogously, the premisses, the global uncertainty, left, ranges from the largest certain value of the left premiss $A_1$ to the smaller certain value of the observation $A_*$.

### III. Interpolating the Position

The linear hypothesis for the position states that the position of the observation $A_*$ and the premisses $A_1$ and $A_2$ are in a linear relationship with coefficient $\alpha$ and that the interpolated conclusion $B_*$ and the conclusions $B_1$ and $B_2$ are also in a linear relationship with the same coefficient $\alpha$. Mathematically, we have:

$$A_1^P = \alpha \cdot A_1^N + (1 - \alpha) \cdot A_2^N$$

$$B_1^P = \alpha \cdot B_1^N + (1 - \alpha) \cdot B_2^N$$

Now, using these two equations we can easily compute the position of the interpolated conclusion $B_*$. First, based on equation 5, we calculate $\alpha$:

$$\alpha = \frac{A_1^P - A_2^P}{A_1^N - A_2^N}$$

Second, using that value in equation 5, we obtain the position $B_*^P$.

### IV. Interpolating the Shape

On Fig. 2 we note that the shape of observation $A_*$ is not necessarily in a linear relationship with the shapes of the premisses $A_1$ and $A_2$, when considered in the position scale (which coincides with the universe of description $X$ as described in section III). One could argue that this lack of linearity depends upon the way the shape is measured, but in fact for any non trivial measure it is easy to imagine a case that breaking the linearity. In fact, a simple way to achieve this is by, first computing the expected shape value (linearly) and then build a fuzzy set observation $A_*$ (the counterexample), which has different shape value. This counterexample exists because the measure is assumed not trivial.

Moreover, this non-linearity applies not only to any general description of the shape but also to any linear shape measure as for instances a length descriptor. Thus, any $\alpha$-cut based method will suffer form this non-linearity.

One of the direct consequences is that methods attempting to linearly interpolate the shape will obtain degenerated shapes [?]. The more recent methods are able to avoid ill solutions, using heuristics that wisely choose among a set of points the ones providing a viable solution. Unfortunately these heuristics are solely designed to avoid degenerative solutions, ignoring any interpolative argumentation.

To overcome the above fundamental reality, we propose to compute the value of the interpolated conclusion by a two step interpolation: first linearly interpolate the expected shape (between the premisses and between the conclusions) and then linearly interpolate the shape deviation between the expected shapes and limit cases, which follow from fundamental assumptions. De facto, our two limit cases correspond, in the one extreme, to zero (assuming that any shape measure is positive), and in the other extreme, to the shape measure of the global uncertainty. The latter corresponds to the assumption that the interpolated conclusion must be at least between the kernels of the conclusions, since they correspond to totally certain rules and since we have a gradual assumption. Notice that this second constraint is closely related to what is called normalisation in other methods.

The solutions are constraint to a reasonable range, corresponding to the limit cases, in which they linearly evolve. The shapes of the premisses and conclusions influence only indirectly the solution via a double linear interpolation: first we interpolated linearly the expected value and second we interpolated linearly the deviation.

#### A. Describing the shape of a trapezoidal fuzzy set

Since, in our description of trapezoidal fuzzy sets (see Section II), the position $A^P$ is an independent variable, the other three parameters (length of the kernel $A^N$, extend of the uncertainties left $A^L$ and right $A^R$) fully and independently characterize the shape. Thus, we propose to decompose the shape interpolation into three separate interpolation applying respectively to the kernels’ length and to the extend of the left and right uncertainties; using as shape measure the length. Notice that, as already mentioned, all the argumentation of Section IV applies not only to a single global shape measure but also to any length descriptor. Thus, the outlined intuition will be used for each of the three interpolation.

#### B. Interpolating the kernel’s length

In order to interpolate the kernel’s length, we perform a double linear interpolation. First, we estimates the linearly expected shapes $A_1^N$ and $A_2^N$ using the kernels’ lengths of the premisses $A_1$ and $A_2$, and of the conclusions’ $B_1$ and $B_2$. As illustrated on Fig. 2 and 3, using simple arithmetic and equation 7 we obtain:

$$A_1^N = (A_1^N - A_2^N) \cdot \alpha + A_2^N$$

$$B_1^N = (B_1^N - B_2^N) \cdot \alpha + B_2^N$$

Second, we linearly interpolate the deviation of the observed kernel $A_1^N$ from the expected kernel $A_1^N$. As we will see below two scenarios appear depending on the relative magnitude of $A_1^N$ with respect to $A_2^N$. It is noteworthy to observe...
that both scenarios cannot be integrated in a single linear transformation.

![Fig. 2](image1.png) **Fig. 2.** The shape of observation $A_*$ is not necessarily in a linear relationship with the shapes of the premisses $A_1$ and $A_2$.

Fig. 3. The shape of the interpolated conclusion $B_*$ should be in the same linear deviation from the expected value, as in the premisses.

1) **Smaller-shape deviation:** Let us assume that $A_N^* < A_E^N$.

In this case the observation is more precise than the expected interpolation. Knowing that any shape measure, and in particular the length kernel, is always positive or null, and using the linear hypothesis for the deviation we have that:

$$A_N^* = \beta \cdot A_E^N + (1 - \beta) \cdot 0 = \beta \cdot A_E^N \quad (10)$$

$$B_N^* = \beta \cdot B_E^N + (1 - \beta) \cdot 0 = \beta \cdot B_E^N \quad (11)$$

Thus, in order to obtain the interpolated conclusion’s kernel-length, we use Equation 10 to obtain:

$$\beta = \frac{A_N^*}{A_E^N} \quad (12)$$

which is then used in Equation 11 to obtain the length $B_N^*$. 

2) **Larger-shape deviation:** It may happen that the observation is less precise than the linearly expected value: $A_N^* \geq A_E^N$.

In that case Equation 10 do not apply, since the observed shape description is out of the linear interpolative range $[0, A_E^N]$. In that case we expect the length of the conclusion to be also larger than the expected linearly interpolated conclusion $B_E^N$, but we claim that its uncertainty can not be larger that the gap between the two kernels. In fact, we know (see Section II) that on the extremes rules 1 and 2 apply, and thus, if something is observed in the middle, the conclusion should also be in the middle (based on the monotonicity of the linear hypothesis). In the case of the kernel’s length calculations the available uncertainty between the premisses $A_1$ and $A_2$ is $U_A^N$, as show on Fig. 1. Consequently, assuming a linear hypothesis for the deviation, we obtain:

$$A_N^* = \gamma \cdot A_E^N + (1 - \gamma) \cdot U_A^N \quad (13)$$

$$B_N^* = \gamma \cdot B_E^N + (1 - \gamma) \cdot U_B^N \quad (14)$$

The same way as in Section IV-B1, in order to obtain the interpolated conclusion’s kernel-length, we use Equation 13 to obtain:

$$\gamma = \frac{U_A^N - A_N^*}{U_A^N - A_E^N} \quad (15)$$

Which is then used in Equation 14 to obtain the length $B_N^*$.

C. **Interpolating the left and right uncertainties**

At this point we know the kernel’s (and fuzzy set) position (Section III) and the kernel’s length (Section IV-B). Now we interpolate, analogously as for the kernel’s length, the left and right uncertainties. We use the exact same equations replacing $N$ by $L$ or by $R$, respectively. The only remarkable change with what have been seen above is the estimation of the global uncertainty left ($U_A^L$ and $U_B^L$) and right ($U_A^R$ and $U_B^R$). As show on Fig. 1 we consider that both global uncertainties ranges from the kernel to kernel, respectively left and right of the observation and interpolated conclusion.

V. INTERPOLATED RESULT IS NON DEGENERATED

Here, I will show that the result is a trapezoidal fuzzy set.
VI. EMPIRICAL COMPARISONS

In this section we compare the proposed interpolation method (the double-linear fuzzy interpolation method, DoLFIn method for short) with some existing well-known methods by [14] (DP method), [3] (BTKY method), and [8] (BRM method).

In the following, we consider three scenarios. First of all, some results obtained when the observation is similar to the premises are presented. Afterwards, a presentation of the obtained results when observation is precise is shown. Finally, obtained results in some particular cases are presented.

Figures Fig. 5 to Fig. 10 should be read as follows. Two columns of different results are presented. Each column is associated with a comparison: the first row shows the two premises and the observation, and the other fourth rows present the conclusion deduced by each of the fourth studied approaches.

A. Observation similar to premises

A first case arises when the observation is similar to the premise. In this case, the observation can be considered as a translation of one premises. It is a very basic case that is often very similar to use case of interpolative reasoning.

B. Precise observation

A second case arises when the observation is precise. It is often the case in real-world application for decision-making process without the fuzzification of the input data.

In Fig. ??, it can be seen that the results here are very similar among the fourth methods. As previously seen, the DP method offers a very general solution that could be hardly linked to the fact that the observation is precise. The BTKY method encounters here a problem due to the connexity of the proposed solution.

In the first column of results, BRM and DoLFIn methods proposed a similar results, but in the second case the results are very different: BRM proposes a more fuzzy solution than the DoLFIn one. This difference is easily explained: BRM is based on the use of the differences between the form of the observation and the forms of the premises to modify the forms of the conclusions. Here, in the first case, the differences are null because the premises are triangular, and thus the construction possesses a kernel which is similar to the conclusion ones. In the second case, the transformation of forms is too strong in the premise space and leads to a bad solution in the conclusion space.

The DoLFIn method offers in the two cases a result that is precise and that takes into account the ratio of kernel sizes between the observation and the premises, adapted to the kernel sizes of the conclusions.
C. Particular observation

In this part, a comparison of the results obtained for two particular cases are shown. The fourth methods present here different results.

As often, the DP method offers a very general solution with some strong counter-intuitive size for the support of the conclusion which is generally too large.

When the premises are precise (first column of results), the BTKY method proposes a solution which is very general. In this case too, the BRM method offers a solution that can be seen a little bit weird in the sense that its support goes outside the scope of the rules. Here, the DoLFin method proposes an interesting solution with a reasonable support size. Moreover, as for the observation and the premises, the solution does not impinge on the kernel of the conclusions of the rules.

When the observation is a precise interval that cover the whole space between the premises (second column of results), DoLFin is the only approach that leads to a conclusion that is a precise interval. In this case, BTKY method does not propose a convex solution, and BRM method constructs a fuzzy set as solution.

VII. CONCLUSION

The conclusion goes here.

REFERENCES

Fig. 10. Particular observation (2)