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Balance Operator: A New Vision on Aggregation Operators.

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Abstract: We introduce a new way to visualize the behavior of aggregation operators based on the analogy with a balance. The use of an analogy permits an intuitive representation of the operator. This metaphor is enough general to include operators of the most common aggregation families. So we obtain a general framework, that allows us to compare the different existing operators. In this paper we will show at first how we establish the analogy, then we will remind which operators are concerned. Subsequently, using the metaphor we will show how to construct the visualization of the behavior. And finally we will discover that the general visualization can be used for particular aggregation analysis.

Introduction

The problem of aggregating fuzzy sets in a meaningful way has been of a central interest since the late 1970s. In most cases, the aggregation operators are defined on a pure axiomatic basis forgetting usually to give a global intuitive vision of the comportment.

That is why, in this paper we present a new way, based on a metaphor, to visualize aggregation operators. The use of a metaphor offers the possibility to illustrate the mathematical and axiomatic choices, providing in this way an intuitive vision of the behavior of the operator.

On this article we will begin with the construction of the analogy, starting from a physical model of a balance and finishing with an aggregation operator: the balance aggregator. After this, we will note that our metaphor is very ample: the balance aggregator is enough general to include operators of the most common aggregation families: t-norms, t-conorm, arithmetic mean, OWA, MICA and Uninorm. That means, not only that we can use it with a lot of classical operators, but also that we obtain a general framework for discussion and comparison. Then we will, employing the analogy, construct the visualization of the behavior of an aggregation operator. We will at first explain how to construct the visualization of the general comportment and then how to add the visualization of the sensibility. We will finally see that we can exploit the last visualizations for the analysis of particular aggregations.

The Balance Framework

The idea of the metaphor consists in establishing a strength relationship between a real world object, on which we have a lot of natural intuition, with an abstract mathematical formula. We will start with a physical model of a real balance and then we will state that the mathematical formula, which computes the total weight, is actually a general form of aggregation operators.

The physical model

The physical model of balance we use is shown in the Figure 1.

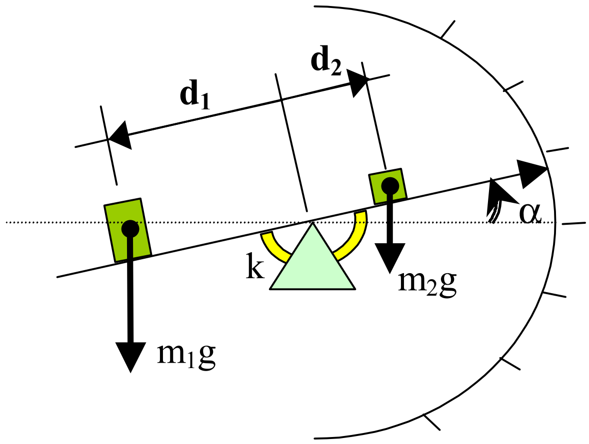

Figure 1: The balance model

The physics laws

Newton's Second Law gives us the following equation:

$$m_1 \cdot \vec{g} \cdot \vec{d}_1 + m_2 \cdot \vec{g} \cdot \vec{d}_2 + k \cdot \alpha = 0 \tag{1}$$

Let be e the position of the fulcrum on the lever, x_i the position of the object i and $w_i = g \cdot m_i$, the weight of the object i, then executing the vector product, and for α close to zero, we obtain:

$$w_1 \cdot \bar{d}_1 + w_2 \cdot \bar{d}_2 + k\alpha = 0 \tag{2}$$

Where $\bar{d}$ is what physicist call the oriented distance[1]:

$$\bar{d}_i = (x_i - e) \tag{3}$$

If we want to read the result of the weight, we only need to know the angle α.. Therefore, we solve for that the equation on α:

[1] $\bar{d}$ is called in physics the oriented distance, because it gives the value of the distance and it is negative when we are on the left of the fulcrum and positive when we are on the right.

$$\alpha = -\frac{1}{k}\left(w_1\overline{d}_1 + w_2\overline{d}_2\right) \tag{4}$$

We can easily generalize this result to n weights:

$$\alpha = -\frac{1}{k}\cdot\sum_{i=1}^{n} w_i\overline{d}_i \tag{5}$$

The balance equation with topology transformation

If we look closer to what we called the oriented distance (3), we will see that it's a function. This function associates to each point x, of the natural scale of the lever, a value of "distance with sign". (See Figure 2)

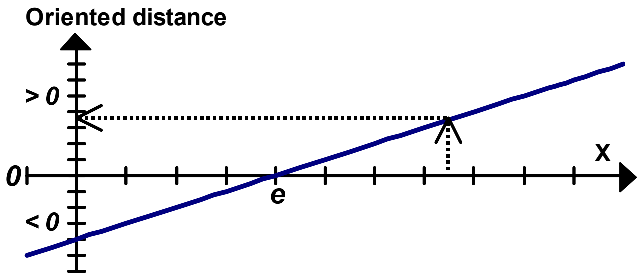

Figure 2: Oriented distance as a function

We observe in this case a linear function, but we can imagine that we want to distort the topology of the scale by using a non-linear function $f : x \rightarrow f(x)$.
Taking this into account the equation of the balance (5) becomes:

$$\alpha = -\frac{1}{k}\cdot\sum_{i=1}^{n} w_i \cdot f(x_1) \tag{6}$$

Let be $y = \sum_{i=1}^{n} w_i \cdot f(x_1)$, then the previous equation can be written:

$$\alpha = -\frac{1}{k}\cdot y \tag{7}$$

We observe here, once again, a linear function. This operator associates to each point y, a value of α.. In other words it's the scale drown on the balance (the weight scale). We can here also decide that the scale won't be linear. If we call h: $y \rightarrow h\ (y)$, the new non-linear function, the balance equation becomes:

$$\alpha = h\left(\sum_{i=1}^{n} w_i \cdot f(x_1)\right) \tag{8}$$

Here α can be understood as the value read on the weight-scale of the balance. In fact it is the transformation by h of the angle between the lever and the horizon. So, what we obtained here, is the formula that computes the total weight from the single weights. And that is exactly the idea of an aggregation operator. We will call this formula (8) the balance equation.

A general form of Classical Aggregation Operators

We observe that the balance equation (8) is a general form of a lot of aggregation operators [1]. Between the most famous we have: the quasi-arithmetic means [2,3,4,8], the archimedian t-norms and t-conorms [9,10], the Ordered Weighted Averaging Operators (OWA) [11,14], symmetric addition on the associative case, the associative compensatory operator C [7] (a general form of Uninorm[5,12]) and finally the Monotonic Identity Commutative Aggregator operator (MICA) [6,13].

The visualization of Aggregation

On this paragraph we will see that the metaphor can be used to visualize the general behavior and the sensibility of an operator of the balance family. But it also can be used for visualization of a particular data aggregation.

The visualization of the general comportment

We already said that the function f associates to each position x, of the natural scale of the lever, a new value, in other words a new position. In order to visualize the action of this functions, we can draw the transformation of a linear scale done by the function f. So we project, using f, a constant step from de the x-axis on the y-axis. In this way we obtain the new-scaled lever on the y-axis. (See Figure 3)

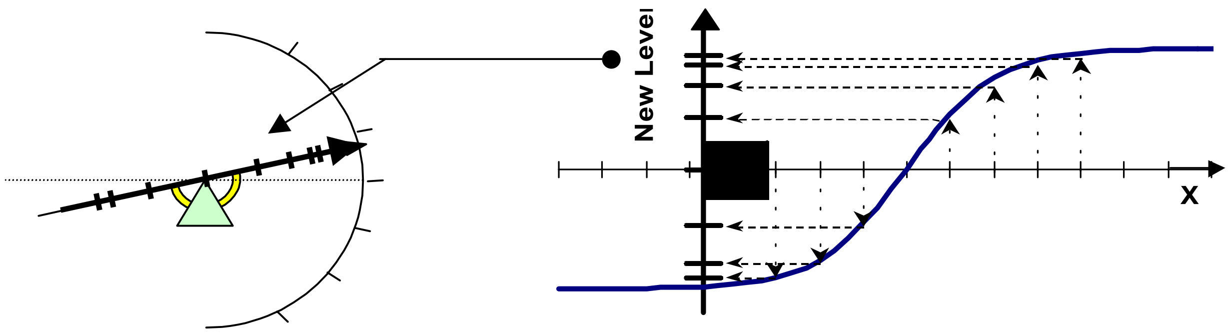

Figure 3. Scale transformation induced by f

If we look now at the function h, we will observe that it transforms the angle between the lever and the horizon into the value read on the weight-scale of the balance. It is also a topology transformation, but the visualization of its action is a bit different. The scale we built with f was construct in order to illustrate the new positioning of arguments separated by a constant step. Here the new scale of weight has to be built in order to show the right total aggregation with a normal functioning of the lever. This last condition is fundamental if we want to preserve our intuition on the balance model. Being precise, what we want is that when the lever makes an angle α with the horizon, the weight shown is $h(\alpha)$. We also expect that the numerical variation of the points appearing on the weight scale remain linear. In other words we want to deform a linear scale of $h(\alpha)$ and put it on the α -space. To do that we use a similar method as for f, but this time we will project a regular step from the y-axis onto the x-axis, and then distort the obtained scale into a part of a circumference (See Figure 4)

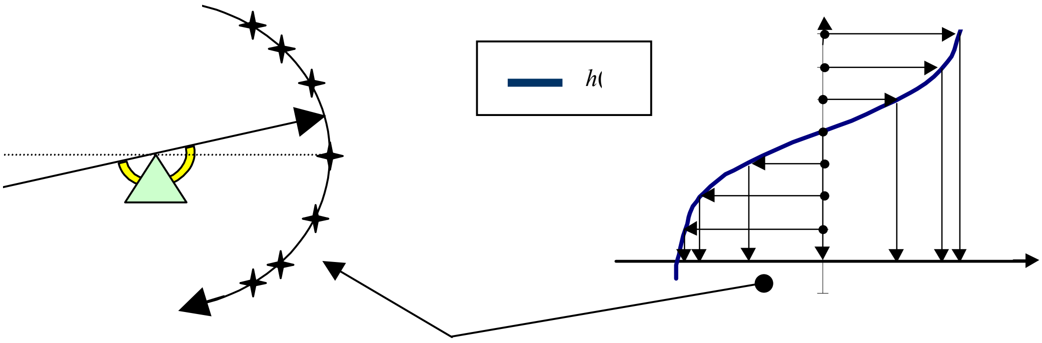

Figure 4: Scale transformation induced by *h*

Visualization of the Sensibility

In order to understand the notion of stability we will focus at first our attention on the Figure 3: Scale transformation induced by *f*. As we already said, the function *f* associates to each position *x,* a new position *f*(*x*). If we analyze more precisely, taking particular attention to the form of the function, we will remark that the more the function *f* is steep, the less the lever scale is serried. That means that a small variation of *x* in a region where *f* is steep will produce an enormous variation in the aggregation. A simply way to quantify this "steepness" and so the sensibility, is to use the derived function of *f*. We will obtain like this the sensibility of the operator to the variation of a particular value of an argument. In an analog way we can study the sensibility of *h,* that will gives us the sensibility of the operator when we obtained a particular aggregated value. We can represent the derived function on our model in order to obtain a general overview (See Figure 5).

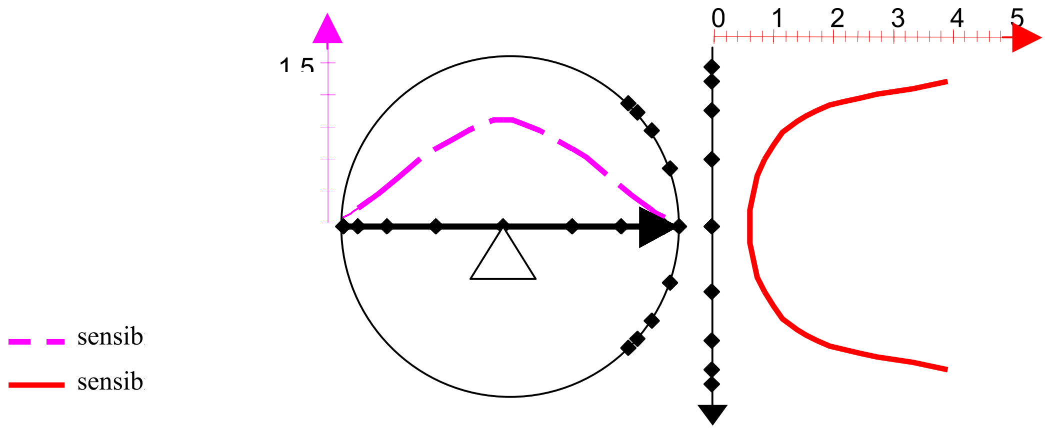

Figure 5: A balance with the sensibility of the lever and of the weight-scale

The visualization of a particular aggregation

We talk of a particular aggregation, when we have a specific set of values to be aggregate, and also their weights, the aggregation formula and therefor the aggregated value. We assume that the aggregation operator can be put on the balance equation form, so that we can construct a balance model, as we indicated before. What we want to prove here is that the positioning of the specific values on the model can be useful in the understanding this particular aggregation. The idea is to place the weights and put the balance on the right aggregation position. In this way we can analyze the resulting physical situation. The weights can be represented by a

rectilinear line with a length proportional to the weight associated to the argument. These objects can be placed with the help of the function *f*, in the same way as explained on the topology transformation paragraph. The fulcrum and the total computed weight give the position of the lever.
We can now draw all the weight that have been used for the particular aggregation, and look on the distribution. We will obtain a graph like this shown on the Figure 6.

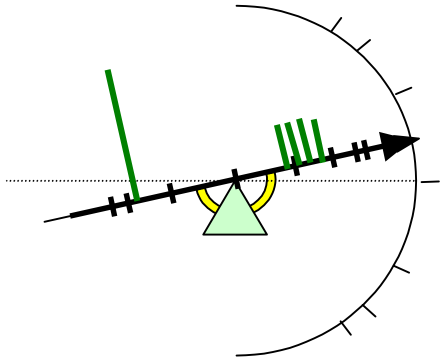

Figure 6: A balance with the sensibility of the lever and of the weight-scale

On this particular example we observe that there is only one argument that is on the left of the fulcrum and all others on the right. We note that the weight of this argument is so important that the lever balanced on to the left. We interpret this particular aggregation as follows: there is big argument against a lot of small arguments. It may be interesting then to take a look closer to this argument, because it can be for example an error.
We can also take a look at the sensibility of the place where the arguments are placed, and in this way we will know if a small variation of this value will change a lot (or not) his position.
We can also look at the sensibility of the area of the total aggregation, and in this way we will know if a small variation of any of the argument will change a lot the computed value.
We remark at this point that all the visualizations done before can be constructed with direct mathematical formulas.

Conclusion

Taking a look at the synonyms of visualization we will find that it means understanding, judgment, appreciation, recognition, and even cognizance, awareness and knowledge. We think that if people have given these meanings to the concept of visualizations, it is because visualization has very often played a fundamental role in understanding, interpreting and solving very different problems. So, we do believe that the visualization should play an important role on the aggregation question.
In this article we started by establishing an analogy between a general aggregation operator and physical model of a balance. Subsequently, we noticed that this general operator is enough general to include operators of the most common aggregation families. After that, we presented how to exploit the established analogy for the use of visualization: We showed how to construct a schema model from the general equation form. Then pushing further the analogy we find that we can also illustrate the sensibility of the operator. Finally we discover that we can use the model, in order to analyze particular data aggregation.
We believe that we have created interesting tools for the understanding of additive-based aggregation. These are results of our metaphor, which seems now to be rich, but not fully investigated. We hope that we will able to exploit it even more.

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