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Aggregating Truth and Falsity Values

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Abstract - *We propose an axiom set for the aggregation of truth values, which leads to the characterization of two truth-aggregation families, a prudent and an enthusiastic. The first one has a cautious attitude choosing between two observed values the one which is more uncertain. The second one has an enthusiastic behavior and will reinforce the result if it observes twice the truth or twice the falsity. When observing falsity and truth the operator gives a compensated value.*
We finish by expounding the use of these operators and their relationship with the traditionally used truth-aggregation operators: the t-norms and t-conorms. Actually the presented operators should be used for the aggregation of different observed truth values for the same phrase vs. the calculus of the truth of a logical phrase.

Keywords: Fuzzy Logic, Uncertainty, Truth Aggregation, Expert Systems, Uninorms.

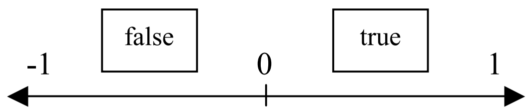
1 Introduction

The problem of aggregating truth values is at the core of the studies in fuzzy logic. But it is to notice that the purpose of this aggregation is to compute the truth value of a logical phrase. Here we are interested in the aggregation of different truth values observed for the same logical phrase.
In the first case, we compute the truth value of a phrase of the type : "the figure *is* quadrilateral" AND "the angle on the left *is* a right angle". Here it is clear that if one statement is true and the other false then the truth value of the whole assertion will be false, which follows the classical logic.
In the second case, we observe two different truth values for the same statement. For example we observe once that "the angle on the left *is* a right angle" is true and then we observe that it is false. This does not mean that the statement is completely false, we just can conclude that we do not know if it is true or false. In this paper we try to give an answer by proposing operators that compute the degree of truth using the two observed degrees.
Before the construction of such operators we define a truth scale, on which we work. Then based on an axiom set we characterize two aggregation families: the *prudent* and the *enthusiastic* one. We conclude by explaining the utility of these operators for real applications.

2 Truth fuzzy set and falsity fuzzy set

Here we are going to assume the truth as a fuzzy set, where 1 is the full truth and 0 represents the zero degree of membership of being true. In the same way we can imagine a fuzzy set for the falsity, but this time we are going to use the negative scale and consider -1 the full falsity and 0 the zero degree of membership of being false. In this way we build a scale of "truth" we can pass progressively from the total falsity to the total truth. So we work on the scale [-1,1] and we will say that [0,1] is the degree of truth and [-1,0] the degree of falsity, where -1 is the full falsity.

Figure 1 : Scale of truth and falsity



In this construction an interesting point is 0. It is the middle point between the full truth and the full falsity. It is actually the null membership to the sets of truth and falsity. So we will consider that it represents the "ignorance", since it is neither true nor false.
We can relate these two fuzzy sets with a negation operator that transforms a truth value into a falsity one with the same degree: $n(x) = -x$. The negation of full truth is full falsity and vice-versa. We also observe that the fixed point of the negation is 0 corresponding to the total ignorance.

3 Basic properties of the operator

On this scale, how can we aggregate the truth values ? In order to build an operator we first dictate some very useful conditions for the aggregation. The operator should be:

- **Monotone:** The decrease of any of the arguments to be aggregated cannot produce an increase in the total value. This condition is related to the Pareto optimum.
- **Commutative:** This property ensures that the aggregation is symmetric, that is, indifferent to the order the arguments to be aggregated are considered.

- **Associative:** we consider that the total result should be independent of the grouping of the arguments. This may be regarded as a criterion of objectivity. This axiom also allows to extend the operator from two arguments to more than two.

The associativity and commutativity taken together allow to avoid some problems inherent to the nature of the systems:

- The architecture of the aggregation: if we aggregate the sources at different levels or steps the associativity and commutativity guarantee that the final result is independent of these levels or steps.
- The temporal aspects of obtaining the data: the order and the size of the parcels received in the time will not have any influence on the total result. We will obtain a final result equal to the aggregation considering that we disposed of all the information from the beginning.
- The computability capacity of the machine executing the aggregation: if the quantity of information is bigger than the memory of the machine processing the aggregation then we will need to do partial aggregations before getting the final result. The associativity and commutativity guarantee that the final result is independent of the method used to make these partial results.

4 The prudent aggregation

Now that we fixed these axioms relative to the general behavior, let us identify some attitudes.
For instance by saying that we want to be *prudent* (*cautious*) while aggregating, let $Prud(x,y)$ denote this operator. We translate this condition as a simple constraint:
Let us imagine that we observe first a full truth and then for the same observation another truth value. What will be the aggregation of this two degrees? If we are *prudent* (*cautious*) we will consider that the final membership should be equal to the non-full truth degree, since we observe first something completely true and then a doubt on this value, a *prudent* attitude is to keep the doubt. Mathematically we translate this by:

$$\text{for all } x \in [0,1] \quad Prud(1,x)=x \qquad (1)$$

A simple choice like this can have many consequences. For instance:
The aggregation of the total ignorance with any truth value equals the total ignorance. Mathematically:

Proposition 1: For all $x \in [0,1]$ $Prud(0,x)=0$.
Proof. Since the aggregation of two truth values should be a truth value ($Prud(0,x)\geq 0$), and using the monotonicity: $0 \leq Prud(0,x) \leq Prud(0,1) = 0$

The *prudent* aggregation of two truth values will always give a smaller (or equal) value than the smallest of any of the values being aggregated. It is exactly what we expect from a *prudent* (*cautious*) aggregation. Mathematically:

Proposition 2: For all $(x,y)\in [0,1]^2$ $Prud(x,y) \leq \min(x,y)$
Proof. Using the monotonicity we have
$Prud(x,y) \leq Prud(x,1) = x$
and by commutativity
$Prud(x,y) = Prud(y,x) \leq Prud(y,1) = y$
Hence, $Prud(x,y) \leq x$ and $Prud(x,y) \leq y$;
that is, $Prud(x,y) \leq \min(x,y)$.

Note: We notice here that the operator actually used on $[0,1]^2$ is a t-norm (see annex).

4.1 Prudent aggregation of the falsity values

Using the negation we can build the operator to be used for the falsity fuzzy set: for all $x,y \in [-1,0]^2$

$$Prud(x,y) = n(Prud(n(x),n(y)) \qquad (2)$$

It is easy to show that this transformation keeps the associativity, the monotonicity and the commutativity.

Other consequences are:

If we observe something completely false and then a doubt on this falsity, the *prudent* aggregation will choose the doubt value. Mathematically:

Proposition 3: For all $x\in [-1,0]$ $Prud(-1,x) = x$
Proof. Obvious using (2).

The *prudent* aggregation of "ignorance" and any falsity value will give full "ignorance".

Proposition 4: For all $x\in [-1,0]$ $Prud(0,x) = 0$.
Proof. Obvious using (2).

The *prudent* aggregation of falsity values will always give a less or equally false value than the less false of the values being aggregated.

Proposition 5: For $(x,y)\in [-1,0]^2$ $Prud(x,y) \geq \max(x,y)$.
Proof. Using (2) and proposition 2.

Note: We notice here that the operator actually used on $[-1,0]^2$ is a t-conorm (see annex) shifted from $[0,1]^2$.

Another way of obtaining the operator to be used on the falsity domain is assuming proposition 3 as an axiom. But, we choose to use the negation for the construction in order to impose exactly the same behavior for the falsity and for the truth.

4.2 Prudent aggregation of falsity and truth values

Let us now take a look at what happens if we aggregate a truth value with a falsity value. In this case we obtain the ignorance, which corresponds to our *prudent* attitude.

Mathematically, we have:

Proposition 6: For $(x,y)\in[-1,0]x[0,1] \cup [0,1]x[-1,0]$

$$Prud(x,y) = 0$$

Proof. Let us assume that $0\le x$ and $y\le 0$. Using the monotonicity we obtain that:

$0 = Prud(0,y) \le Prud(x,y) \le Prud(x,0) = 0$

and for $x\le 0$ and $0\le y$, we have

$0 = Prud(x,0) \le Prud(x,y) \le Prud(0,y) = 0$

A consequence of the precedent result is that the *prudent* operator is continuous on $[-1,1]^2$. (of course if and only if the underlying t-norm is continuous). In fact if the underlying t-norm is continuous, then the operator is continuous on $[-1,0]^2 \cup [0,1]^2$. Proposition 6 shows that the operator on $[-1,0]x[0,1] \cup [0,1]x[-1,0]$ equals 0 and using proposition 1 and 4, we have the continuity on $[-1,1]^2$.

This property is interesting since it translates the fact that the variation of the aggregated value do not jump from a value to another. It gives some stability to the result.

The general behavior: The associativity of the global operator was showed in [3]. We can resume the *prudent* aggregation on the following figure, where T is a t-norm:

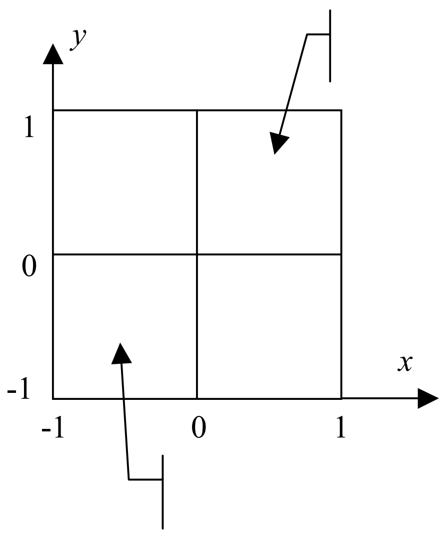


Figure 2 : *Prudent* aggregation

Interpretation: We showed in proposition 1 that when aggregating two truth values the result value will be a truth value smaller than the smallest of the values to be aggregated.

When aggregating two falsity values, proposition 2 applies, which means that the result is always bigger that the maximum of the values. And the aggregation of two falsity values will be a smaller falsity value than any of the falsity values to aggregate.

As we already said in proposition 6, the aggregation of a truth value with a falsity value will give the total "ignorance".

Summing up, the general tendency of this operator is to converge to the "ignorance". If we have two truth values we will compute an aggregated value that is closer to the total ignorance than any of the two initial values.

5 The enthusiastic aggregation

We studied in detail the construction of a *prudent* operator, let us now take a look at the *enthusiastic* aggregation. Let us denote by $Enthu(x,y)$ such an aggregation operator. Of course we still request the operator to be associative, monotone and commutative. But this time we will impose that if we observe the total ignorance and a truth or a falsity value, then enthusiastically we will use this value as the final aggregation.

Mathematically we have

$$\text{For all } x \in [-1,1] \quad Enthu(0,x) = x \qquad (3)$$

The choice of these characteristics for the operator has the following consequences:

The *enthusiastic* aggregation of the total truth with any other truth value will be the total truth. Mathematically :

Proposition 7: For all $x\in[0,1]$ $Enthu(1,x) = 1$.

Proof. Using the monotonicity we have

$$1 = Enthu(0,1) \le Enthu(x,1) \le 1$$

The *enthusiastic* aggregation of the total falsity with any other falsity value will be the total falsity:

Proposition 8: For all $x\in[-1,0]$ $Enthu(-1,x) = -1$.

Proof. Using the monotonicity we have

$$-1 \le Enthu(-1,x) \le Enthu(-1,0) = -1$$

We have a reinforcement when observing two truth values. That is to say that the observation of two truth values will be aggregated into a value bigger than the biggest of the observed values. Mathematically :

Proposition 9: For all $(x,y)\in[0,1]^2$ $Enthu(x,y) \ge \max(x,y)$

Proof. Using the monotonicity we have $x = Enthu(x,0) \le Enthu(x,y)$ and, by commutativity $Enthu(x,y) = Enthu(y,x) \ge Enthu(y,0) = y$.

Hence, $Enthu(x,y) \ge x$ and $Enthu(x,y) \ge y$; that is, $Enthu(x,y) \ge \max(x,y)$.

The same way, we have a reinforcement when observing two falsity values. We obtain a bigger falsity result than what was observed. Mathematically :

Proposition 10: For all $(x,y)\in[-1,0]^2$ $Enthu(x,y)\le \min(x,y)$

Proof. Since $(x,y)\in[-1,0]^2$ and using the monotonicity we have $Enthu(x,y) \le Enthu(x,0) = x$ and, by commutativity $Enthu(x,y)=Enthu(y,x) \le Enthu(y,0) = y$

Hence, $Enthu(x,y) \le x$ and $Enthu(x,y) \le y$; that is, $Enthu(x,y) \le \min(x,y)$.

When observing truth and falsity we have a compensatory behavior. In fact the final value will be closer to the total ignorance than any of the observed values. Mathematically:

Proposition 11: For all $(x,y)\in[-1,0]x[0,1] \cup [0,1]x[-1,0]$
$\min(x,y) \le Enthu(x,y) \le \max(x,y)$
Proof. Let assume that $y \le 0 \le x$. Using the monotonicity we obtain that:
$\min(x,y) = y = Enthu(0,y) \le Enthu(x,y)$,
and $Enthu(x,y) \le Enthu(x,0) = x = \max(x,y)$

Note: We remark that we actually have a t-conorm on $[0,1]^2$ and on $[-1,0]^2$ we observe a t-norm shifted from $[0,1]^2$.

In order to obtain a similar behavior on the falsity and truth domain, we can request the operator to be self-involutive with respect to the negation. Mathematically we want:

For all $x,y \in [-1,1]^2$ $Enthu(x,y)= n(Enthu(n(x),n(y))$

If we reason in terms of t-norms and t-conorms, what we want is that these two operators with the negation form a De Morgan triplet.

5.1 Enthusiastic aggregation of truth and falsity

At this point we characterized the behavior of our operator on $[0,1]^2$ and on $[-1,0]^2$, which corresponds to the aggregation of truth with truth or of falsity with falsity. But how to aggregate falsity and truth, so that the operator keeps its properties on $[-1,1]^2$? An immediate solution is to use a construction similar to the ordinal sum [7]. This method consists in using a min or a max on $[-1,0]x[0,1] \cup [0,1]x[-1,0]$. We obtain, this way, two operators, a *negative enthusiastic* and a *positive enthusiastic*.
The *negative enthusiastic* will use the min in order to aggregate the truth with the falsity, and it will always choose the falsity over the truth. Figure 3 represents this kind of operator, where S is a t-conorm.
The *positive enthusiastic* will use the max and will always choose the truth over the falsity.
These two families seem to provide a nice aggregation operator. But if we take a look closer at the *negative* and *positive enthusiastic* operators, we will notice that they have an annoying characteristic. These operators are not continuous around the total ignorance.
For instance, a *negative enthusiastic* aggregation of a very true value with a quasi ignorance (but true) or with a quasi ignorance (but false), will give a very different result. In the first case, because of proposition 9 we know that the aggregation will be bigger that the maximum of the two true values (i.e. a very true value close to 1).

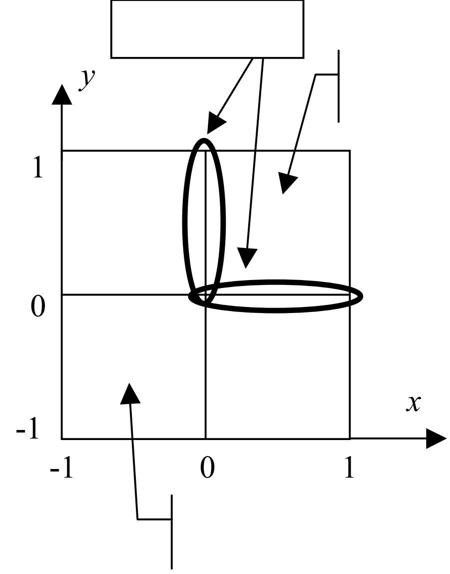


Figure 3 : Negative enthusiastic aggregation

In the second case, because of the construction we will use the min, which will give a quasi ignorance but false, (i.e. a falsity value close to 0). So the result jumps for these two very close observations from a value close to 1 to a value close to 0. Mathematically this is translated by a discontinuity on $\{(x,0) / x\in]0,1]\}$ and on $\{(0,x) / x\in]0,1]\}$.

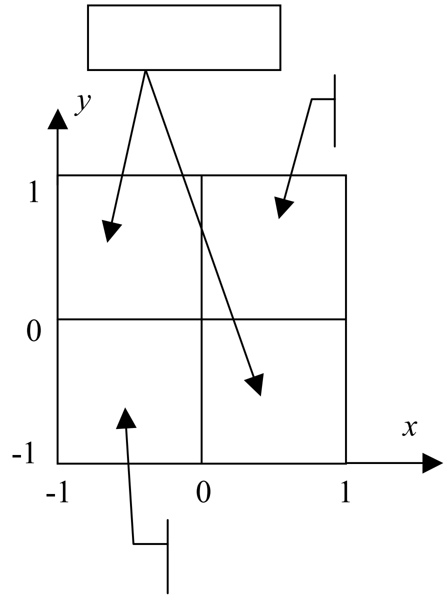


Figure 4 : Discontinuity on negative enthusiastic aggregation

Even if this characteristic is not suitable for the aggregation it is clearly the consequence of the choice of our attitude. We are *enthusiastic*, so if we observe two truth values we take the best one, forgetting about the value close to the total ignorance. But if this quasi ignorance becomes a falsity value then, since we have a *negative* attitude we will choose the falsity over the truth.

We may think that the discontinuity is related to *negative* attitude of the operator. But unfortunately we observe the same problem for the *positive enthusiastic* operators. This time the aggregation of a total falsity value and two different quasi ignorance values will give very different results. In fact, we are *enthusiastic*, so if we observe two falsity values we take the certain one (i.e. a very falsity value close to -1), forgetting about the value close to the total ignorance (see property 10). But if this quasi ignorance becomes a truth value, then since we have a *positive* attitude we will choose the truth over the falsity.
Mathematically this is translated by a discontinuity on $\{(x,0) \setminus x\in[-1,0[\}$ and on $\{(0,x) \setminus x\in[-1,0[\}$.

5.2 Continuous enthusiastic aggregation

Since these discontinuities can be annoying, we can built a continuous operator using a method inspired by the representation theorem of the continuous archimedean t-norms and t-conorms. In concrete terms we will base the construction of the continuous enthusiastic operator on the use of an additive generator h. Given that on $[0,1]^2$ the enthusiastic operator equals a t-conorm, our generator h should on [0,1] equal to a generator g of a t-conorm. Mathematically we have that :

For all $x \in [0,1]$ $h(x) = g(x)$

where the function $g : [0,1] \to [0,+\infty]$ is continuous, strictly increasing, with $g(0) = 0$ and $g(1) = +\infty$

In order to obtain a similar behavior on the falsity domain, we use the negation to obtain the generator on [-1,0] :

For all $x \in [-1,0]$ $h(x) = n(g(n(x))) = - g(-x)$

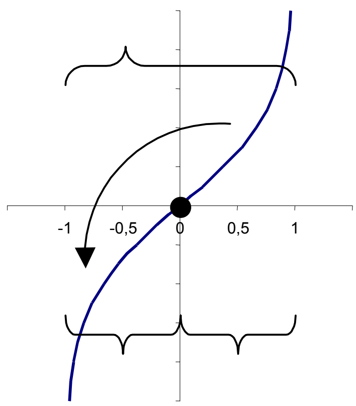


Figure 5 : Enthusiastic aggregation generating function

We obtain a function $h : [0,1] \to]-\infty,+\infty[$, which is continuous, strictly increasing bijection, symmetric with respect to the point (0,0) and verifying $h(0) = 0$. The continuous enthusiastic operator being defined by :

$$Enthu(x,y) = h^{-1}(h(x)+h(y)) \qquad (4)$$

We verify that we have an *enthusiastic* operator, since it is by construction associative, commutative and for all $x \in [-1,1]^2$ $Enthu(0,x) = x$ (because $h(0)=0$).
Automatically we inherit the properties 8,9 and 10, which translate the reinforcement behavior when observing twice the truth (or twice the falsity) and compensation when confronting truth and falsity.

A particularity of this family is that when observing the total truth and then any other value (even if it is false), we will enthusiastically take the certain value (the total truth) as the aggregated value. Mathematically :

Proposition 12: For all $x \in]-1,1]$ $Enthu(x,1) = 1$.
Proof. Obvious taking into account that $h(1)=g(1)=+\infty$

The same happens, when we observe the total falsity and any other value, then we choose the total falsity.

Proposition 13: For all $x \in [-1,1[$ $Enthu(x,-1) = -1$.
Proof. Obvious taking into account that

$$h(-1) = - g(-(-1)) = - g(1) = - \infty$$

Under the light of propositions 12 and 13 a natural question arises: what happens when confronting the total truth with the total falsity? Actually formula (4) does not give any answer. Mathematically we observe that (4) is continuous on $[-1,1]^2$ besides the limit points (0,1) and (1,0), where the function is undefined. This translates the impossibility of giving a value when observing a total truth and a total falsity.

Representation Theorem: We have shown that our construction gives an operator with a certain number of remarkable properties, but it is interesting to notice that it has been shown in [1] and [6] that if we request our operator to be associative, monotone, commutative, self involutive, with a neutral element and continuous on $[-1,1]^2$ except on the points (-1,1) and (1,-1), then it can be written under the form (4).

This result shows in particular that if we want an operator verifying the basic properties enunciated at the beginning, having the same behavior for the truth and for the falsity (self involutive) and enthusiastic with respect to the total ignorance, then the only continuous solution is the one presented here.

6 Applications

At this point we characterized two operators devoted to the aggregation of truth values. But actually, the aggregation of truth values is usually being done by conjunctive operators (t-norms) or disjunctive operators (t-conorms). So, what is the difference between these operators and the introduced construction ? The main difference is that we are not interested in the logical truth value of a logical formula, but in the aggregation of different values of truth for the same proposition. So, in order to compute the truth value of a logical formula of the form "a is A (degree of truth x) AND b is B (degree of truth y)", we use of course a t-norm T : "(a is A AND b is B) (degree of truth $T(x,y)$)". But if we observe first "a is A (degree of truth x)" and then we observe "a is A (degree of truth y)", then we will use a *prudent* or *enthusiastic* aggregation to obtain "a is A (degree of truth $Prud(x,y)$)" or "a is A (degree of truth $Enthu(x,y)$)".

The aggregation presented in this paper should be used whenever we observe different truth values for the same proposition. A good example when this arrives is the case of the expert systems, where several rules point to the same conclusion. In fuzzy systems it is common to use in this case a t-norm (or a t-conorm), but actually the operator that should be used is a *prudent* or *enthusiastic* operator. Actually, if we examine the handling of uncertainty in non-fuzzy expert systems, we notice that the heuristics used to aggregate the truth values are nothing else than operators of the families presented in this paper. For instance the medical expert systems MYCIN [2] used a continuous *enthusiastic* operator [11], based on the probabilistic t-conorm.

Obviously, the use of these operators is not restricted to the expert systems. Their use is suitable in other fields as sensor fusion and decision making. More generally in the case of data fusion, we recommend to use these operators when several sources (expert, sensors, etc.) give different certainty values for the same fact (object, statement, etc.).

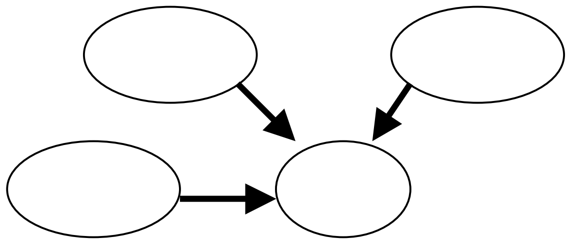


Figure 6 : Multiple Source Aggregation

Another interesting case were the prudent and/or the enthusiastic operators are suitable is the case of data received in a temporal way. Then the revision of the already received data should be performed with the proposed operators. We notice here that the associativity becomes particularly interesting.

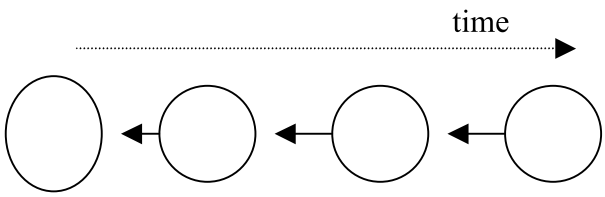


Figure 7 : Temporal Data Aggregation

7 Conclusion

This paper is a first attempt at the formalization of the aggregation of truth values in a non logical calculus way. With necessary conditions (axioms) we characterized two truth aggregation families: the *prudent* and the *enthusiastic*. The first one has a *cautious* attitude choosing between two observed values the one that is more uncertain. The second one has an *enthusiastic* behavior and will reinforce the result if we observe twice the truth or twice the falsity. When observing falsity and truth the operator gives a compensated value.

We finished by expounding that these operators should be used for the aggregation of different truth values observed for the same phrase vs. the calculus of truth of a logical phrase.

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Annex

The t-norms and the t-conorms: The concept of a triangular norm was introduced by Menger[8] in order to generalize the triangular inequality of a metric. The current notion of a t-norm and its dual operation (t-conorm) is due to Schweizer and Sklar [9,10]. Both of these operations can also be used as a generalization of the Boolean logic connectives to multi-valued logic. The t-norms generalize the conjunctive 'and' operator and the t-conorms generalize the disjunctive 'or' operator. This situation allows them to be used to define the intersection and union operation in fuzzy logic. This possibility was first noted by Höhle [12], Klement [13] and Alsina, Trillas, and Valverde [14] very early appreciated the possibilities of this generalization. Bonissone [15] investigated the properties of these operators with the goal of using them in the development of intelligent systems. T-norm and t-conorms have been well-studied and very good overviews and classifications of these operators can be found in the literature, see [16] and [17].

Definition of a t-norm:

A t-norm is a function T : [0,1]x[0,1] $\rightarrow$ [0,1], having the properties of commutativity, associativity, monotonicity and 1 as neutral element.

The most common examples are :

- Zadeh t-norm: min(x,y)
- The probabilistic t-norm: $x.y$
- Lukasiewicz t-norm: max(x+y-1,0)

Definition of a t-conorm:

A t-norm is a function S : [0,1]x[0,1] $\rightarrow$ [0,1], having the properties of commutativity, associativity, monotonicity (increasing) and 0 as neutral element.

The most common examples are :

- Zadeh t-conorm: max(x,y)
- The probabilistic t-conorm: $x + y - x.y$
- Lukasiewicz t-conorm: min(x+y ,1)